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SOME INTERESTING SOLUTIONS OF DIFFERENTIAL EQUATIONS

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The equation $\int f(x)dx = F(x) + C$ where $\dfrac{dF(x)}{dx} = f(x)$ is familiar to us when studying integration. For example $\displaystyle\int x^2 dx = \frac{x^3}{3} + C$. When evaluating an integral we are in effect solving a differential equation. A differential equation in two variables x and y is an equation which contains x or y or both and at least one derivative of one of these two variables with respect to the other. For example:

$$x\frac{dy}{dx} + 4x^2y = 3x + 4$$

$$\frac{dy}{dx} + y = -1$$

$$\frac{dy}{dx} = x^2 + 1$$

When evaluating $\int x^2 dx$ we are in effect solving the differential equation $\dfrac{dy}{dx} = x^2$ obtaining $y = \dfrac{x^3}{3} + C$ where C is a constant of integration.

We are living in a changing world. In any system whether living, chemical, mechanical, commercial, biological etc., there are many variables changing as a result of changes in other variables. So we have great opportunities to study such systems by applying $\dfrac{dy}{dx}$ (which may be interpretted as the rate of change of the variable y with respect to the variable x) which gives rise to differential equations. We then have the problem of solving these differential equations. Some of these solutions turn out to be very interesting. We give three such examples:

EXAMPLE 1 The Age Of A Person.

The age A of a person varies with time t. If A and t are measured in the same units of time (say years) the associated differential equation is:

$$\frac{dA}{dt} = 1$$

Integration gives:

$$A = t + C.$$

If further information is given, for example, if the date of birth of the person is 1912 then we can determine C since $A = 0$ when $t = 1912$ and so $0 = 1012 + C$ giving $C = -1912$. So $A = t - 1912$. We now know the age of this person (while alive) as a function of time. The solution (or trajectory) of this differential equation is a straight line.

EXAMPLE 2 Freely Falling Bodies.

Galileo Galilei (1614-1642) was able to conclude that a falling stone did so with constant acceleration. That is for a freely falling stone (Fig.1) we have

$$\frac{d^2x}{dt^2} = g$$

where x is displacement and g, the acceleration due to gravity, is constant.

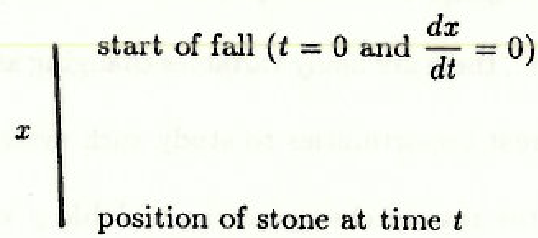

Fig.1

Newton (1642-1717, note that he was born the year Galileo died) deduced that $\frac{d^2x}{dt^2} = g$ by applying calculus and his law of universal gravitation

$$F = G\frac{m_1m_2}{r^2}$$

where F is the force between the two masses m_1 and m_2 with a distance r apart and $G(= 6.672 \times 10^{-11} Nm^2kg^{-2})$ is a universal constant. Using Newton's second law $F = ma$ we obtain:

$$\frac{GmM}{r^2} = m\frac{d^2x}{dt^2}$$

where m is the mass of the stone, M is the mass of the earth and r is the distance between the stone and the centre of the earth.

Rearranging we obtain:

$$\frac{d^2x}{dt^2} = \frac{GM}{r^2}$$

During the fall of the stone the distance r is practically constant and is considered equal to the radius R of the earth. So

$$\frac{d^2x}{dt^2} \simeq \frac{GM}{R^2} = g.$$

The solution is easily verified to be

$$x = ut + \frac{1}{2}gt^2$$

where u is the initial velocity. In this case $u = 0$ since it falls from rest. So

$$x = \frac{1}{2}gt^2$$

In this case the trajectory (solution) of the differential equation is a parabola.

EXAMPLE 3 Simple Harmonic Motion.

The differential equation associated with simple harmonic motion is

$$\frac{d^2x}{dt^2} = \ddot{x} = -\omega^2 x \tag{1}$$

where x is the displacement and ω is a constant. It has solution

$$x = E\cos\omega t + F\sin\omega t \text{ where } E \text{ and } F \text{ are constants.}$$

Nevertheless it turns out to be worthwhile to introduce as a 2nd variable $y = \frac{dx}{dt}$ (or velocity). At an immediate level this is natural to do because knowledge of (x, y) at any specific time determines the constants E and F and hence $x(t)$ for all times t. (In physics a particle is determined by its position and momentum at any time).

If we multiply equation (1) by $\dot{x}$ we obtain

$$\dot{x}\ddot{x} + \omega^2 x\dot{x} = 0$$

Integration then gives:

$$\frac{(\dot{x})^2}{2} + \frac{\omega^2 x^2}{2} = C$$

Putting $y = \dot{x}$ we obtain:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \qquad (2)$$

where

$$A^2 = \frac{2C}{\omega^2} \text{ and } B^2 = 2C \text{ (What are } A \text{ and } B \text{ in terms of } E \text{ and } F\text{?)}$$

Equation (2) represents the ellipse in Fig.2 with semimajor axis A and semiminor axis B.

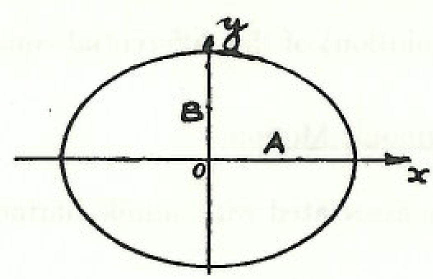

Fig.2

If $\omega^2 = 1$ then the ellipse becomes a circle and in any case the trajectory of the differential equation representing simple harmonic motion is a closed curve.

So let's continue studying the sustained oscillations in a real system with the closed curves appearing in the solutions of the differential equations associated with the system. Historically this first arose when studying the motion of the earth about the sun. This motion is considered to be in a state of sustained oscillations (sometimes referred to as a stationary motion). Kepler (1571-1630) ventured in his first law of planetary motion that planets move in elliptical orbits having the sun as one of the foci. Halley (1656-1742) (of Halley's comet fame) was the first in 1675 to derive the first proof (geometric) that the sun was at the focus of elliptical planetary orbits. We start with simple harmonic motion since it is the simplest and it is known to give a closed curve as a solution. We can replace the (2nd order) equation

$$\frac{d^2x}{dt^2} = -x \tag{3}$$

with a (1st order) system by setting $y = \dfrac{dx}{dt}$.
Since $\frac{d^2x}{dt^2} = \frac{dy}{dt} = x$ the equation (3) is equivalent to the system

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x\end{aligned} \tag{4}$$

System (4) is a special case of the system

$$\begin{aligned}\frac{dx}{dt} &= X(x,y) = ax + by\\ \frac{dy}{dt} &= Y(x,y) = cx + dy\end{aligned} \tag{5}$$

which we can easily check corresponds to

$$\frac{d^2x}{dt^2} - (a+d)\frac{dx}{dt} + (ad-bc)x = 0 \tag{6}$$

Such "linear" systems can be completely solved and essentially only 6 types of trajectories can occur. This is further evidence that introducing the variable $y = \frac{dx}{dt}$ was a good idea. The following solutions (Fig.3) are possible:

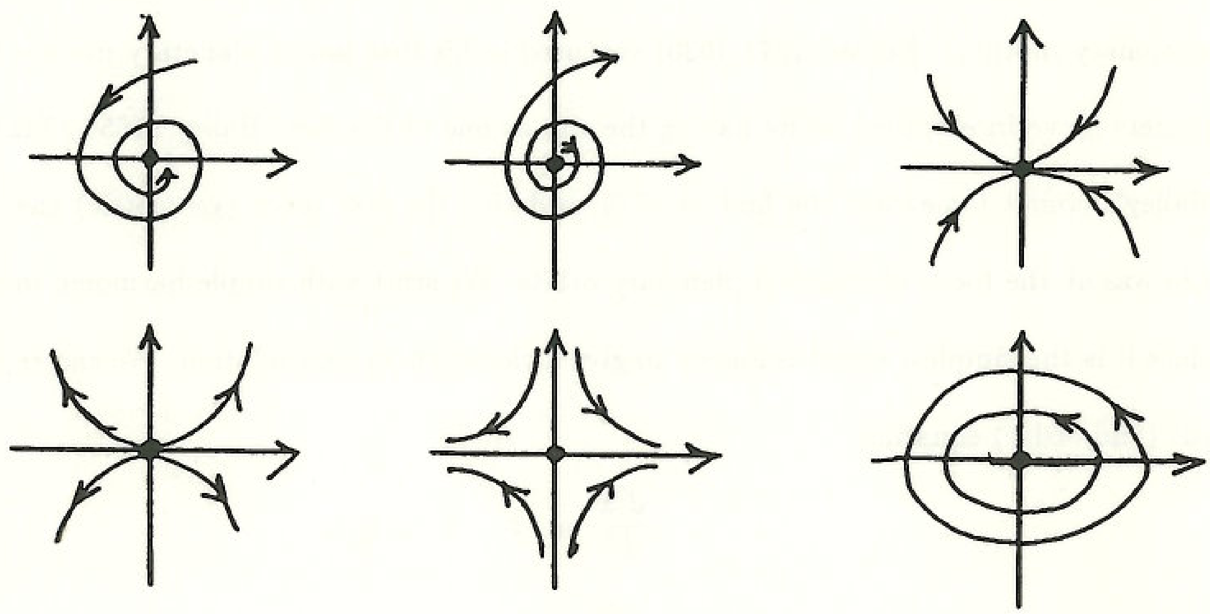

Fig.3

The xy-plane here is called the phase plane. Those points in the phase plane at which $\frac{dx}{dt} = \frac{dy}{dt} = 0$ are called singular points (stable focus, unstable focus, stable node, unstable node, saddle point and centre are different types of singular points) and they correspond to equilibrium points in physics. Although we are not in a position to verify that the only solutions possible are of the types listed above it is easy to check that the system

$$\begin{aligned}\frac{dx}{dt} &= -x + y \\ \frac{dy}{dt} &= -y - x\end{aligned} \tag{7}$$

has, as two solutions, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t}\cos t \\ -e^{-t}\sin t \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-t}\sin t \\ e^{-t}\cos t \end{pmatrix}$.

These equations give rise to the 1st spiral type trajectories listed above. Perhaps the key observation in checking this is to note that the distance from the origin $\sqrt{x^2+y^2}$ equals e^{-t} (which tends to zero as t grows indefinitely).

As we have implied the only case of the above system that gives a closed trajectory (solution) is the centre where:

$$\frac{dx}{dt} = by$$

$$\frac{dy}{dt} = cx$$

$$\text{and } \frac{d^2x}{dt^2} = -\omega^2 x \quad \text{assuming } bc \text{ is negative}$$

which is the case of the simple harmonic motion (as we established in example 3). Note that there is no net energy exchange between the system and the surroundings during one complete oscillation. But this does not stop us applying simple harmonic motion to the motion of the earth around the sun as there are no frictional forces involved and so there is no net energy exchange in one revolution. However we do know that there are systems on the earth exhibiting sustained oscillations despite the inevitable presence of frictional forces. One such example is the pendulum clock which is in a state of sustained oscillations when working. Energy is given to this system to sustain the oscillations. For many purposes it is inappropriate to approximate such systems by a model based on simple harmonic motion. Consequently in the years since Galileo mathematicians have had to discover other ways of studying sustained oscillations.