PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of Parabola; your solution(s) may be used if they are received in time.

Q.762 Simplify the sum

$$\sec\theta + \sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \cdots + \sec(n-1)\theta\sec n\theta$$

$$= \sum_{k=0}^{n} \sec(k-1)\theta\sec k\theta$$

Q.763 Take the usual decimal representation of any whole number, x. If the number of digits is odd, place a zero at the beginning (e.g. replace 2350002 by 0235002). Break it up into consecutive pairs of digits, and let $T(x)$ be the sum of all these two-digit whole numbers.

e.g. $T(2350002) = 02 + 35 + 00 + 02 = 39$.

(i) Show that for any $x, x - T(x)$ is a multiple of 99, and deduce that $T(x)$ and x leave the same remainders on division by 9, and also on division by 11.

(ii) If $x = 1989^{1989}$, show that $T(x) < 400000$ and calculate $T(T(T(x)))$ exactly.

Q.764 We have n points in the plane, no two distances between points being equal. A straight line interval is drawn connecting each of the points to the nearest of the others. Prove that no point lies on more than 5 of the intervals.

Q.765 If n is any whole number, let $d(n)$ denote the number of different factors of n, and let $\sigma(n)$ denote their sum. For example, the factors of 12 are 1,2,3,4,6 and 12, so $d(12) = 6$ and $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. If the factorisation of

n into prime numbers is $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$

show that $d(n) = (1 + a_1)(1 + a_2) \cdots (1 + a_k)$.

and $\sigma(n) = \left(\frac{p_1^{a_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{a_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_k^{a_k+1}-1}{p_k-1}\right)$

[Since $12 = 2^2 \times 3$, these formulas give $d(12) = (2+1) \times (1+1) = 6$

and $\sigma(12) = \left(\frac{2^3-1}{2-1}\right)\left(\frac{3^2-1}{3-1}\right) = 7 \times 4 = 28$.]

Q.766 Both $d(n)$ and $\sigma(n)$ obviously vary greatly from one value of n to the next. (For example, compare $d(n)$ and $\sigma(n)$ when n is 359 or 360). But if one averages the values of $d(n)$ or $\sigma(n)$ for $n = 1, 2, \cdots, N$ a less chaotic behaviour can be observed. Let $[x]$ denote the greatest integer not larger than x;

e.g. $[8] = [8.34..] = 8$

(a) Show that $\sum_{n=1}^{N} d(n) = \sum_{k=1}^{N} [\frac{N}{k}]$

and $\sum_{n=1}^{N} \sigma(n) = \sum_{k=1}^{N} k[\frac{N}{k}]$.

(b) Deduce that $\frac{N}{2} < \frac{1}{N}\sum_{n=1}^{N} \sigma(n) \leq N$

and that $2 - \frac{1}{N} \leq \frac{1}{N}\sum_{n=1}^{N} d(n) \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$ for $N \geq 2 < \log N + 1$ (using the result of Q.768.)

Q.767. It was first proved by L. Euler that the limit sum of the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

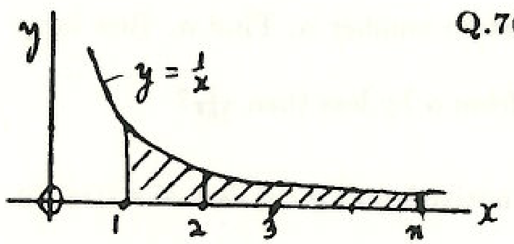
is equal to $\frac{\pi^2}{6}$.

Assuming this, calculate the sum of

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots+\frac{1}{(2n-1)^2}+\cdots$$

and of $\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\frac{1}{19^2}+\frac{1}{23^2}+\frac{1}{25^2}+\cdots$

where there remain on the denominators the squares of whole numbers which are relatively prime to 6.



Q.768. Let $S_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ and let A denote the area in the first quadrant lying under the graph of $y=\frac{1}{x}$ between $x=1$ and $x=n$. Show that $A < S_n < A+1$. (By calculus, it is shown that $A=\int_1^n \frac{1}{x}dx = \log_e n$.)

Q.769. There are 25 pupils in a class. Of these 18 take French, 14 take History and 9 take Biology, but only one takes all 3 of those subjects. English is a compulsory subject and everyone taking French, History or Biology gained a grade of either B or C in the English exam. There were 5 pupils in the class whose English mark was a D.

How many in the class obtained an A for English?

How many take both History and Biology?

Q.770 The distance to the sun is about 387 times the distance to the moon. Find approximately the ratio of the volume of the sun to that of the moon.

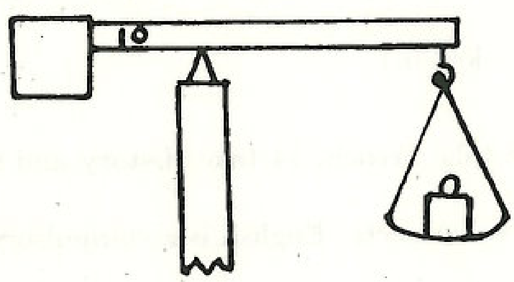
Q.771 Consider the sequence of ordered pairs of whole numbers (1,2); (4,9); (17,38); (72,161); (305,682); $\cdots$ Denoting the nth pair by (x_n, y_n), the construction rule is $(x_1, y_1) = (1, 2)$.

$$\left.\begin{aligned} x_{n+1} &= 2x_n + y_n \\ y_{n+1} &= 5x_n + 2y_n. \end{aligned}\right\} \text{ for all } n \geq 1.$$

(i) Prove that $x_n \geq 4^{n-1}$ for all n.

(ii) Find a formula for $5x_n^2 - y_n^2$ for all n.

(iii) When n is large $\frac{y_n}{x_n}$ is very close to a certain number α. Find α. How large should n be taken to ensure that $\frac{y_n}{x_n}$ differs from α by less than $\frac{1}{10^6}$?



Q.772 A primitive balance consists of a straight length of material attached at one end to a heavy block. At the other end is a scale pan suspended from a cuphook. A mark labelled 0 is placed on the rod at the balance point when there is no weight on the scalepan. How can one calibrate the instrument if one has only a single one-kgm weight, a tape measure and a pen.