SOLUTIONS OF PROBLEMS 753-761

Q.753 When multiplying two whole numbers a student by mistake reduced the tens digit in the answer by 7. He checked his answer by dividing it by the smaller factor, obtaining the quotient 48 and the remainder 17. Find the two factors.

ANSWER: Let x, y be the given whole numbers, $x > y$, and let p be the fallacious answer obtained for the product.

Then $xy = p + 70$ and $p = 48y + 17$

Thus $(x - 48)y = 87$. Since $y > 17$ the only factorisations of 87 to consider are 1×87 and 3×29. However $x - 48 = 1, y = 87$ contradicts $x > y$, so only $x - 48 = 3, y = 29$ yields a permissible solution, $(x, y) = (51, 29)$.

Q.754 A bath was being filled with water from two taps. The cold tap was first turned on for a period equal to one fifth of the time required to fill the bath by the hot tap alone. Then it was turned off and the hot tap was run for one-fifth of the time required to fill the bath by the cold tap alone. By now the bath was 5/12 full, and it required another 5 minutes to complete filling the bath with both taps turned on.

Find how long each tap separately takes to fill the bath.

ANSWER: Let the cold tap fill the bath in c minutes and the hot tap fill the bath in h minutes.

The given information translates into

$$\frac{h}{5c} + \frac{c}{5h} = \frac{5}{12} \tag{1}$$

and

$$\frac{5}{c} + \frac{5}{h} = \frac{7}{12} \tag{2}$$

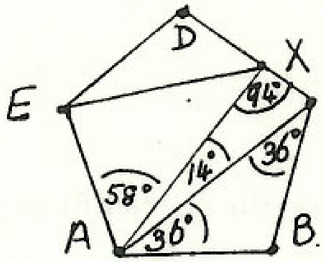
(1) becomes $12(h^2 + c^2) = 25ch$, and then $12(h + c)^2 = 49ch$ (3)

and (2) becomes $60(h + c) = 7ch$ (4)

From (3) and (4) one obtains easily $h + c = 35$ and $hc = 300$ (5)

and after an easy calculation one obtains $(h, c) = (20, 15)$ or $(h, c) = (15, 20)$.

Hence one tap fills the bath in 15 minutes, the other in 20 minutes.



Q.755 $ABCDE$ is a regular pentagon, and X is the point on CD such that $X\hat{A}B = 50°$. Calculate $D\hat{E}X$.

ANSWER: Let the side length of the regular pentagon be 1 unit.

Since $A\hat{B}C = 108°, C\hat{A}B = A\hat{C}B = 36°$ and $^*AC = 2\cos 36°$.

In $\Delta ACX,^* AX = \sin A\hat{C}X \frac{^*AC}{\sin A\hat{X}C} = \sin 72° \frac{2\cos 36°}{\sin 94°}$, by the sine rule.

By the cosine rule in ΔEAX,

$$^*EX^2 =^* AE^2 +^* AX^2 - 2^*AE^*AX \cos E\hat{A}X$$
$$= 1 + 4\frac{\sin^2 72° \cos^2 36°}{\sin^2 94°} - 4\frac{\sin 72° \cos 36°}{\sin 94°} \cos 58°. \qquad (1)$$

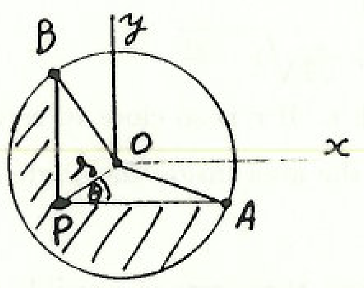
Now by the sine rule,

$$\sin A\hat{E}X =^* AX \frac{\sin E\hat{A}X}{^*EX} = \frac{2\cos 36° \sin 72° \sin 58°}{\sin 94°{}^*EX} \qquad (2)$$

where $^*EX$ is obtained as the square root of the RHS of (1).

Finally $\theta = A\hat{E}D - A\hat{E}X = 108° - \sin^{-1}\left(\frac{2\cos 36° \sin 72° \sin 58°}{\sin 94°{}^*EX}\right)$

Calculation gives $^*EX = 1.32087, A\hat{E}X = 82.06°, \theta = 15.94°$.



Q.756 The figure shows a circle, centre 0, radius 1. P is an interior point and A, B are points on the circumference such that $A\hat{P}B = 90°$ and such that exactly half the area of the disc lies within the angle.

(i.e. the area shaded $= \frac{1}{2}\pi$). Let $^*OP = r$ and $O\hat{P}A = \theta$.

(i) Find the range of possible values of r.

(ii) For r in that range, find an expression relating r and θ.

ANSWER:

Choose cartesian axes Ox, Oy parallel to PA, PB respectively (see **Fig.1**)

The coordinates of P, A, and B are $(-r\cos\theta, \ -r\sin\theta)$,

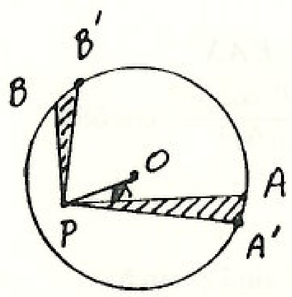
$(\sqrt{1-r^2\sin^2\theta},\ -r\sin\theta)$,

and $(-r\cos\theta,\ \sqrt{1-r^2\cos^2\theta})$ respectively.

Note that $x\hat{O}B = \cos^{-1}(-r\cos\theta)$, and $x\hat{O}A = \sin^{-1}(-r\sin\theta)$ so that $A\hat{O}B = \cos^{-1}(-r\cos\theta) + \sin^{-1}(+r\sin\theta)$

$\therefore$ Area of region $APBC$ = area of sector AOB+ area $\triangle AOP$+ area $\triangle BOP$

$$A(\theta) = \tfrac{1}{2}1^2(\cos^{-1}(-r\cos\theta)+\sin^{-1}(r\sin\theta))+\tfrac{1}{2}r\sin\theta(\sqrt{1-r^2\sin^2\theta}+r\cos\theta)$$
$$+\tfrac{1}{2}r\cos\theta(\sqrt{1-r^2\cos^2\theta}+r\sin\theta) \qquad (1)$$



(i) It is obvious enough from the figure that, for fixed r, $A(\theta)$ is greatest when $\theta = \frac{\pi}{4}$. [If one desires a supportive analytic argument, note that when $\theta < \frac{\pi}{4}, \cos\theta > \sin\theta$ and

$$AP = \sqrt{1-r^2\sin^2\theta} + r\cos\theta > \sqrt{1-r^2\cos^2\theta} + 2\sin\theta = BP.$$

When θ increases by a very small $\delta\theta$, the corresponding change δA in $A(\theta)$ is given by area APA'− area BPB' in Fig.2 where $A\hat{P}A' = B\hat{P}B' = \delta\theta$. Neglecting second order terms, $\delta A = \frac{1}{2}AP^2\delta\theta - \frac{1}{2}BP^2\delta\theta > 0$.

Hence as θ increases from 0 to $\frac{\pi}{4}, A(\theta)$ increases. Since clearly $A(\theta) = A(\frac{\pi}{2}-\theta), A(\frac{\pi}{4})$ is the maximum value of A.]

Note that $A(\frac{\pi}{4}) = \frac{1}{2}(\frac{\pi}{2} + 2\sin^{-1}(\frac{r}{\sqrt{2}})) + \frac{r^2}{2} + \frac{r}{\sqrt{2}}\sqrt{1-\frac{r^2}{2}}$.

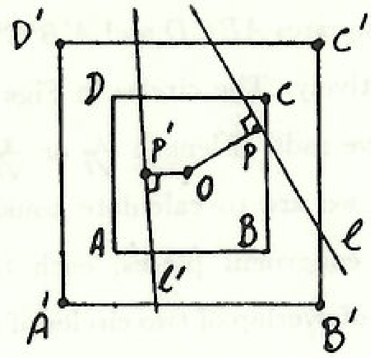
For $0 \le r < 1$, this expression increases with r. If r is so close to zero that $A(\frac{\pi}{4}) < \frac{1}{2}\pi$, there is no value of θ such that the area inside the angle is half that of the disc.

If r_o is the value of r such that $A(\frac{\pi}{4}) = \frac{\pi}{2}$ then the range of possible values of r is given by $r_o \le r < 1$. (We must have $r < 1$ since P is inside the circle). Solving the equation by a numerical method, I obtained $r_o = 0.48256$.

(ii) Provided r is in the above range, there will be a corresponding value of θ such that $A(\theta) = \frac{\pi}{2}$. Then (1) gives

i.e. $\pi = \cos^{-1}(-r\cos\theta) + \sin^{-1}(r\sin\theta) + r\sin\theta\sqrt{1-r^2\sin^2\theta} +$
$$r\cos\theta\sqrt{1-r^2\cos^2\theta} + r^2\sin 2\theta$$

This is the desired relation between r and θ.



Q.757 The figure exhibits two squares $ABCD$ and $A'B'C'D'$ with a common centre 0, and with AB parallel to $A'B'$. A line ℓ cuts across the squares but does not intersect any of the intervals AA', BB', CC' or DD'. P is the foot of the perpendicular from 0 to ℓ.

(Two possible positions of ℓ and P are shown in the figure).

(i) Describe the region which is the locus of P.

(ii) Calculate its area in terms of the side lengths of the two squares.

ANSWER: (i) On Fig.1 are shown circles on diameters OA and OA'. Of the five parallel lines, ℓ_2 and ℓ_4 pass through A and A' respectively, and ℓ_3 intersects AA'; ℓ_5 does not cut across the squares. Hence if perpendiculars from 0 meet the lines at P_1, P_2, P_3, P_4 and P_5, only P_1 is in the desired locus. Note that P_2 and P_4 lie on the circles and P_3 is between them. It follows that the locus of P consists of the unshaded pieces inside the large square in Fig. 2, obtained by deleting (shading) the region between the circles in Fig.1, and similar regions at the other corners.

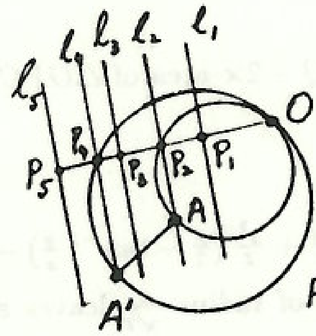


Fig 1

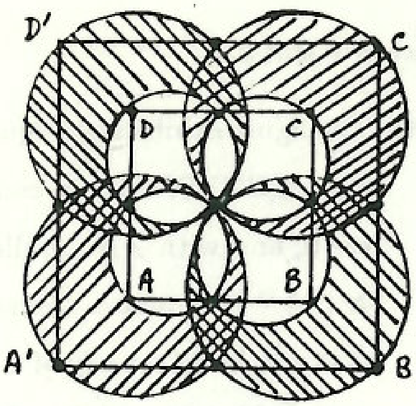


Fig 2

(ii) Let $2x, 2y$ be side lengths of the squares $ABCD$ and $A'B'C'D'$ respectively. The circles in Figs 1 and 2 have radii of length $\frac{x}{\sqrt{2}}$ or $\frac{y}{\sqrt{2}}$. The area we are to calculate consists of four congruent pieces, each the region of overlap of two circles of radius $\frac{x}{\sqrt{2}}$ which intersect orthogonally; and another four congruent pieces, each consisting of what is left of a circle of radius $\frac{x}{\sqrt{2}}$ when two overlap regions with circles of radius $\frac{y}{\sqrt{2}}$ have been removed.

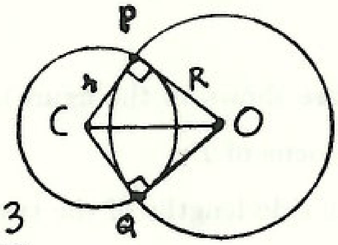


Fig 3

The overlap area of circles of radii r and R which intersect at right angles is given by (refer to Fig.3)

$A =$ area of sector POQ + area of sector $PCQ - 2\times$ area of $\triangle OPC$

$= r^2 \tan^{-1}\frac{R}{r} + R^2 \tan^{-1}\frac{r}{R} - rR.$

When $r = R = \frac{x}{\sqrt{2}}$, this area is $\frac{x^2}{2}\left(\frac{\pi}{2} - 1\right)$ (*)

When $r = \frac{x}{\sqrt{2}}, R = \frac{y}{\sqrt{2}}$, the result is $\frac{x^2}{2}\tan^{-1}\frac{y}{x} + \frac{y^2}{2}\left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right) - \frac{xy}{2}$

The removal of two such pieces from a circle of radius $\frac{x}{\sqrt{2}}$ leaves an area

$(x^2 - y^2)\left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right) + xy$ (#)

Thus the required area is 4 times the sum of the areas given by (*) and (#), namely $(\pi - 2)x^2 + (x^2 - y^2)(2\pi - 4\tan^{-1}\frac{y}{x}) + 4xy$.

Q.758 Let $f(x) = x^n$ where n is a fixed positive integer and $x = 1, 2\cdots$ Define a by its decimal expansion $a = 0.f(1)f(2)f(3)\cdots$ For example if $n = 5$ the first few digits in a are $a = 0.13224310243125\cdots$ using the digits in order of $1^5, 2^5, 3^5, 4^5, 5^5, etc.\cdots$ Is it possible to choose n so that a is a rational number?

ANSWER: Since a is defined by a non-terminating decimal expansion, it is a rational number only if that expansion recurs. Suppose there are p digits in the recurring block and that the decimal starts to recur at the mth decimal place.

Let N be a multiple of 10^p exceeding m.

Then $f(N) = N^n$, a number ending with pn zeros, all of which will appear in a block after the mth place in the constructed decimal. Since this is obviously impossible, we are forced to conclude that the constructed decimal does not recur, and hence that a must be irrational.

Q.759 A function $f(n)$ is defined for positive integers n in such a way that $f(1) = 1, f(2) = 2$, and if $3^{m-1} \leq q < 3^m$ and $r = 0,1$ or 2 then $f(3q+r) = r3^m + f(q)$.

For how many values of n between 1788 and 1988 is $f(n)$ equal to n?

ANSWER: Let $n = a_k3^k + a_{k-1}3^{k-1} + \cdots + a_1 3^1 + a_0$ where each a_j is 0,1 or 2, except that $a_k \neq 0$. Then $n = 3q + r$ where $r = a_0$ and $q = a_k3^{k-1} + a_{k-1}3^{k-2} + \cdots + a_1$. Since $3^{k-1} \leq q < 3^k$ we have $f(n) = a_0 3^k + f(q)$.

Repeating this process, we obtain eventually

$$f(n) = a_0 3^k + a_1 3^{k-1} + \cdots + a_{k-1}3^1 + f(a_k).$$

Since $a_k = 1$ or 2, $f(a_k) = a_k$.

It is now clear that $f(n) = n$ if and only if n when expressed in "ternary form" (i.e. using 3 as the base of the number system instead of the more usual ten) is a "palindromic" number, unaltered by reversing the order of the digits.

Now $(1788)_{10} = (2110020)_3$ and $(1988)_{10} = (2201122)_3$.

All the palindromic ternary numbers between them, in increasing order, are 2110112, 2111112, 2112112, 2120212, 2121212, 2122212, 2200022 and 2201022. Hence there are eight dates in our history since Captain Phillip for which $f(n) = n$.

Q.760 The real numbers x and a are related by $\sqrt{x^2 - a^2} + 2\sqrt{x^2 - 1} = x$. Find the range of possible values of a and solve for x when a is in that range.

ANSWER: Since $\sqrt{x^2 - a^2} \geq 0, 2\sqrt{x^2 - 1} \leq x$, whence $3x^2 \leq 4$.

For $\sqrt{x^2 - a^2}$ to be real, we must have $a^2 \leq x^2$. So there are no real solutions unless $a^2 \leq \frac{4}{3}$.

Squaring $\sqrt{x^2 - a^2} = x - 2\sqrt{x^2 - 1}$ yields after simplification

$4x\sqrt{x^2 - 1} = 4x^2 - 4 + a^2$.

Squaring again, and collecting terms yields

$$(16 - 8a^2)x^2 = a^4 - 8a^2 + 16 = (4 - a^2)^2$$

Hence the (unique) solution is given by $x = \dfrac{4 - a^2}{2\sqrt{4 - 2a^2}}$, for any a such that $a^2 \leq \frac{4}{3}$.

Q.761 (i) Let p denote the perimeter of a triangle, and r the radius of the inscribed circle. Prove that $r \leq \frac{p}{6\sqrt{3}}$.

(ii) $P_1, P_2 \cdots P_n$ are the vertices of an n-gon inscribed in a circle. The centre O of the circle lies inside the n-gon. Let A denote the sum of the areas of the circles inscribed in the n triangles $\Delta P_1OP_2, \Delta P_2OP_3, \cdots \Delta P_nOP_1$, and let B denote the area of the n-gon. Show that $\frac{A}{B} \leq \frac{\pi}{3\sqrt{3}}$. If $0 < k < \frac{\pi}{3\sqrt{3}}$ show that for any $n > 3$ there exists such a polygon with $\frac{A}{B} = k$.

ANSWER: (Note that we have corrected the last inequality from $n \geq$ to $n > 3$)

(i) From fig. 1, it is clear that

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Correct Solutions. P.S. Maylott (Sydney Grammar School) sent a correct solution of Q.753, and most of Q.761.

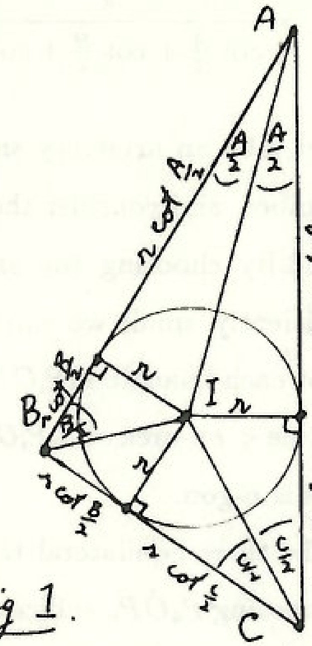


fig 1.

$$p = 2r\left(\cot\frac{\hat{A}}{2} + \cot\frac{\hat{B}}{2} + \cot\frac{\hat{C}}{2}\right)$$

$$= 2r\left(\frac{\sin\left(\frac{\hat{A}}{2}+\frac{\hat{B}}{2}\right)}{\sin\frac{\hat{A}}{2}\sin\frac{\hat{B}}{2}} + \cot\frac{\hat{C}}{2}\right)$$

$$= 2r\left(\frac{2\sin\left(\frac{\hat{A}}{2}+\frac{\hat{B}}{2}\right)}{\cos\left(\frac{\hat{A}}{2}-\frac{\hat{B}}{2}\right)-\cos\left(\frac{\hat{A}}{2}+\frac{\hat{B}}{2}\right)} + \cot\frac{\hat{C}}{2}\right)$$

$$2r\left(\frac{2\cos\frac{\hat{C}}{2}}{\cos\left(\frac{\hat{A}}{2}-\frac{\hat{B}}{2}\right)-\sin\frac{\hat{C}}{2}} + \cot\frac{\hat{C}}{2}\right)$$

(since $\frac{\hat{A}}{2}+\frac{\hat{B}}{2}+\frac{\hat{C}}{2}=\frac{\pi}{2}$).

Since $\cos\left(\frac{\hat{A}}{2}-\frac{\hat{B}}{2}\right) \leq 1$, the minimum value of $\frac{p}{r}$ for any given $\hat{C}$ is obtained when $\hat{A}=\hat{B}$ (and it is equal to $2\left(\frac{2\cos\frac{\hat{C}}{2}}{1-\sin\frac{\hat{C}}{2}} + \cot\frac{\hat{C}}{2}\right)$).

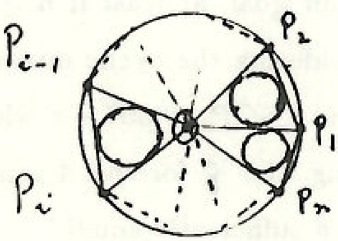


fig 2.

By the symmetry of the R.H.S. of (i), for $\frac{p}{r}$ to be minimum we must similarly have $\hat{B}=\hat{C}$. i.e. all of $\hat{A}, \hat{B}$, and $\hat{C}$ must be equal to $\frac{\pi}{3}$, whence $\frac{p}{r} \geq 2\times 3\cot\frac{\pi}{3} = 2\times 3\sqrt{3}$.

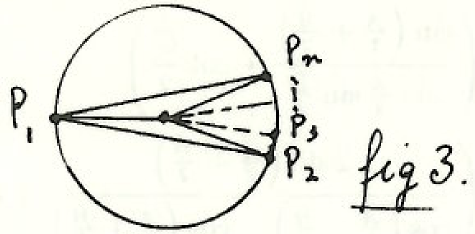
(ii) Hence, for any triangle, $\frac{\text{area of incircle}}{\text{area of triangle}} = \frac{\pi r^2}{\frac{1}{2}pr} = \frac{2\pi r}{p} \leq \frac{\pi}{3\sqrt{3}}$. Therefore, in Fig.2, for each triangle $\Delta P_{i-1}0P_i$, area of incircle $\leq \frac{\pi}{3\sqrt{3}}\times$ area of $\Delta P_{i-1}0P_i$.

Adding, $A =$ sum of areas of incircles $\leq \frac{\pi}{3\sqrt{3}}$ sum of areas of Δ's $\leq \frac{\pi}{3\sqrt{3}}B$.

To do the last part it will be enough to show that for any $\epsilon > 0$, however small, there exists an n-gon for which $\frac{A}{B} < \epsilon$, and one for which $\frac{A}{B} > \frac{\pi}{3\sqrt{3}} - \epsilon$. For, by starting with either of these n-gons and sliding the vertices round the circle until they coincide with the vertices of the other one, the value of $\frac{A}{B}$ will vary continuously and must attain every value between ϵ and $\frac{\pi}{3\sqrt{3}} - \epsilon$ at least once in the process.

Note that if any angle, $\hat{C}$ say, of a triangle is sufficiently small, $\cot\frac{\hat{C}}{2}$

is sufficiently large so that $\frac{\text{area of incircle}}{\text{area of triangle}} = \frac{2\pi r}{p} = \frac{\pi}{\cot\frac{\hat{A}}{2} + \cot\frac{\hat{B}}{2} + \cot\frac{\hat{C}}{2}}$ can be made as small as desired.



Now let ϵ be an arbitrary small positive number, and consider the n-gon in Fig.3.By choosing the angle $P_2\hat{O}P_n$ sufficiently small we can ensure that for each triangle ΔP_iOP_{i+1} area of incircle $< \epsilon\times$ area of ΔP_iOP_{i+1}.

Adding these n inequalities yields $A < \epsilon \times B$ for this n-gon.

In Fig.4, the sum of the areas of the incircles of the three equilateral triangles is equal exactly to $\frac{\pi}{3\sqrt{3}}\times$ area $P_1P_2P_3P_4$. By choosing $P_4\hat{O}P_n$ sufficiently small, both the additional area $P_1P_4\cdots P_n$ and the sum of the areas of incircles of the small triangles $\Delta P_4OP_5, \cdots, \Delta P_nOP$, can be made as small as desired, so that $\frac{A}{B}$ for the n-gon $P_1\cdots P_n$ can be made to differ from $\frac{\pi}{3\sqrt{3}}$ by an arbitarily small amount. This achieves our goal, at least if $n \geq 5$. The case $n = 4$ can similarly be handled by considering the cyclic quadrilateral in Fig.5, in which $P_1\hat{O}P_2 = P_2\hat{O}P_3 = 60°$ but $P_3\hat{O}P_4 = 60° + \theta$ where θ is small. You will have no difficulty in observing that $\frac{A}{B}$ for this 4-gon can be made as close as desired to $\frac{\pi}{3\sqrt{3}}$ by choosing θ sufficiently small.

(However, for $n = 3$, it is not possible to construct an n-gon with $\frac{A}{B}$ close to $\frac{\pi}{3\sqrt{3}}$. Hence the correction indicated at the beginning is necessary.)