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CUBIC EQUATIONS (AND A LITTLE HISTORY)

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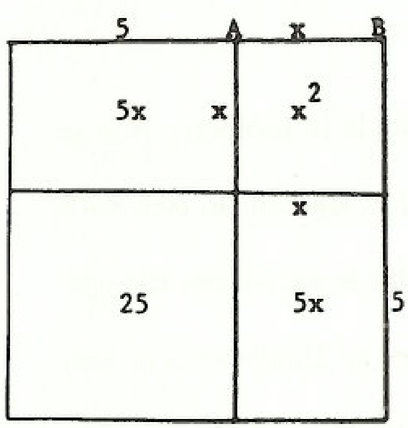
Historians seem to generally agree that the first appearance of something which is recognizably civilization occurred in the southern part of Mesopotamia about 5000 years ago. The various peoples who concurrently or successively occupied this area around and between the Tigris and Euphrates are popularly called Babylonians, and it was they who largely laid the foundations of mathematics. Most Babylonian mathematics was developed from about 2100BC, and a fundamental problem of older Babylonian algebra (expressed using modern notation) is: "find a number x which, when added to its reciprocal yields a given number b." That is, solve $x + \frac{1}{x} = b$ or $x^2 - bx + 1 = 0$. The instructions they gave for solving such a problem were:

form $\left(\frac{b}{2}\right)^2$; then $\sqrt{\left(\frac{b}{2}\right)^2 - 1}$ and then $\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - 1}$, $\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - 1}$.

Of course the Babylonians didn't have this modern notation and they considered only concrete examples but, nevertheless, it would be churlish not to admit that they knew how to solve the quadratic equation.

Given the truly gigantic advances that the ancient Greeks made in geometry it is at first surprising that they didn't achieve anything really substantial in algebra. In fact, with the notable exception of Diophantus of Alexandria (c250AD), who is sometimes thought of as a "Hellenized Babylonian", little was done (see the reference to Diophantus in Bob Hart's article on Fermat's last conjecture which appeared in our last issue). Possibly Diophantus' work would have lead to more significant breakthroughs if the old Greek world at Alexandria had managed to survive as a centre of learning for longer than it did.

Although Diophantus is sometimes called "the father of algebra" the man who is generally given this title comes from the Arabic world. He is Mohammed ibn Musa al-Khowarizmi (c780-c850); his name "ibn Musa al-Khowarizmi" means "the son of Musa, of Khowarizm". (The city of Khowarizm is now called Khiva and is in the Uzbek Soviet Socialist Republic.) More relevant to us is that the word "al-Khowarizmi" yielded the word "algorism" when transliterated in Europe in about the 12th century. ("Algorism" eventually became "algorithm" in the 18th century.) Al-Khowarizmi wrote several books on mathematics and astronomy of which his "Al-jabr" or, to use its full-title, "Hisab al-jabr wal-muqabala" is the most important. The effect of the book on the western world can be easily appreciated since its title has given us the word "algebra". It seems that "al-jabr" meant something like "restoration" or "completion" apparently referring to the transposition of subtracted terms to the other side of an equation. (The Moors' influence in Spain is found in **Don Quixote** where the word "algebrista" is used for "bone-setter" or "restorer".)



In his **Al-jabr** al-Khowarizmi considers quadratic equations in some detail (as did Diophantus before him). In it he solves (our notation) $x^2 + 10x = 39$. Let $AB = x$ and construct on it the square as indicated. Then the area of the total square is $39 + 25 = 8^2$. Therefore its side must be 8, and so $x = 3$.

There are many comments that one can make about al-Khowarizmi's solution which at least shows how the term "completing the square" arose. We make just three. Firstly, the above notation is not al-Khowarizmi's (even the Hindu- Arabic numerals $0, 1, 2 \cdots, 9$ which the book introduced to Europe were of a different shape, and the equality symbol had yet to be invented; "=" was introduced by the Englishman Robert Recorde in 1557). Next, like the Babylonians and the Greeks he eschewed negative numbers (whatever notation we use, an equation of the form $x^2 + 10x - 39 = 0$ is not a possibility). Finally observe how 39 is to be implicitly interpreted as square units (i.e., units of area) and 10 as linear units. (The elimination of dimensional considerations in algebra didn't occur until the time of Descartes and the appearance of his "Discourse on Method" in 1637!) Given these remarks it shouldn't be too hard to believe that we are not in a very strong position from which to attack the cubic. Nevertheless a number of Arab mathematicians tried very hard to solve it and of them Omar Khayyam (1048-1131) is the most famous. Omar wrote his own "al-jabr" which was used as a text in Persian schools until the 19th century, and he thought the cubic couldn't be solved "algebraically". (Despite Omar Khayyam's achievements as a mathematician/astronomer/philosopher it is for his poetry that he is best remembered - Edward Fitzgerald's translation of the Rubaiyat made him famous in the west and a significant proportion of mathematicians can quote some of its quatrains.)

In fact we have to wait another 400 years for the solution to the cubic. It takes place far from the civility and orderliness of Omar's world in the hurly-burly of Renaissance Italy. Although it is difficult to be totally precise here it is thought that the big breakthrough was made by Scipione del Ferro (c1465-1525) who was a professor of mathematics at Bologna. Del Ferro seems to have found a way to solve the equation

$$x^3 + ax = b \quad \text{where } a, b > 0.$$

(our notation again) but we have only indirect information about his solution. It is believed that he entrusted his solution to a pupil Antonio Maria Fior who proceeded to live off it by challenging others to contests at mathematical problem solving, in 1535 he was unlucky enough to challenge Niccolo Tartaglia (c1506- 1559). The arrangements of the "duel" were that each contestant posed the other 30 problems and that each then submitted his solutions to a notary at the end of 50 days:

"These are the thirty problems posed by me Antonia Maria Fior to you Master Niccolo Tartaglia.

1. Find me a number such that when its cube root is added to it the result is six, that is 6.

   ⋮

30. There are two bodies of 20 triangular faces whose corporeal areas added together make 700 braccia, and the area of the smaller is the cube root of the larger. What is the smaller area?"

Each of the 30 problems is reducible to the problem of solving particular cubics of Del Ferro's kind. Just 8 days before his deadline expired Tartaglia found the method and solved all 30 problems. Surprisingly Fior did not manage to solve a single one of the 30 problems put to him by Tartaglia – surprising for one could be solved by del Ferro's formula! Before continuing it's perhaps worthwhile commenting a little on Tartaglia since he's generally cast as a romantic figure in mathematics. During his childhood Tartaglia was badly wounded and left for dead when his native city – Brescia – was captured by the French. Somehow he managed to survive his dreadful wounds but his larynx had been slashed so that he was always to have difficulty speaking. As a result he was given the nickname "Tartaglia" meaning "stutterer". Although he left school before he learnt to write his name he eventually translated Archimedes and Euclid into Italian, and wrote 2 books (in Italian) essentially motivated by problems in artillery.

Tartaglia turned down various requests to reveal his method but under much emotional pressure he eventually told Gerdamo Cardano or Cardan (1501-1576). This was in 1539 in Milan, and was divulged as a rhyme to Cardano only after Tartaglia had extracted a pledge from Cardano that he would never reveal the secret. (Tartaglia used a rhyme as an aid to memory; it has to be remembered that he had very little useful notation.) Cardano is generally portrayed as the villain in the story but as we will see he did have a crucial role to play in the development of mathematics. Before explaining what this was a comment on Cardano's personality seems appropriate. If Cardano had have imagined how he would be remembered by posterity undoubtedly he would have thought as a brilliant doctor; in his autobiography he compares himself to such medical figures as Hippocrates and Galen. To be fair he was one of the most famous doctors of his time and perhaps he was an outstanding diagnostician but whether this skill was enhanced by his belief in magic, premonitions, demons and his own supernatural ability is not so clear. The pope himself used his astrological predictions but this was only after Cardano had extricated himself from jail where he had been sent for heresy (for constructing a horoscope for Christ). His energy carried over into his mathematics and scientific work and he published a number of books on mathematics of which two were very important: **Liber de Ludo Aleae (The Book on Games of Chance)** and **Ars Magna (The Great Art)**. The former is essentially the first book on probability theory (in it Cardano even gives advice as to how to cheat), whilst the "great art" of the latter is the art of al-Khowarizmi.

Ars Magna appeared in 1545, and it contained Tartaglia's solution of the cubic (and much more). Tartaglia was furious but there was little he could do. In reality Cardano had done mathematics a great service. The publication of **Ars Magna** is frequently taken as the beginning of the modern period in mathematics for a great number of important algebraic concepts comes from **Ars Magna**. Cardano's significance in the history of mathematics is determined most of all not by his specific achievements (he did not have many) but by the fact that in **Ars Magna** he saw the path along which algebra would develop.

At last we consider the general cubic

$$y^3 + \alpha y^2 + \beta y + \gamma = 0$$

Suppose we let $y = x - \frac{\alpha}{3}$. Then

$$(x - \frac{\alpha}{3})^3 + \alpha(x - \frac{\alpha}{3})^2 + \beta(x - \frac{\alpha}{3}) + \gamma = 0$$

$$\text{so that } x^3 - 3x^2\frac{\alpha}{3} + 3x(\frac{\alpha}{3})^2 - (\frac{\alpha}{3})^3 + \alpha(x^2 - 2\frac{\alpha}{3}x + (\frac{\alpha}{3})^2) + \beta(x - \frac{\alpha}{3}) + \gamma = 0$$

$$\text{or } x^3 + x(\frac{\alpha^2}{3} - \frac{2\alpha^2}{3} + \beta) - (\frac{\alpha}{3})^3 + \frac{\alpha^3}{3^2} - \frac{\alpha\beta}{3} + \gamma = 0$$

$$\text{or } x^3 + ax + b = 0 \qquad (1)$$

This shows it's enough to be able to solve a cubic without the "square term", and this was understood by Cardano in **Ars Magna**. We'll now solve (1) and explain, at least to some extent, at the completion of the proof the major technical/psychological difficulties that Tartaglia, Cardano et al had had to face. The proof we present follows only the idea of the argument in **Ars Magna**; Cardano argued geometrically in terms of volumes.

In (1) we let $x = u - v$. Then

$$(u-v)^3 + a(u-v) + b = 0$$

$$\text{or} \quad u^3 - 3uv(u-v) + a(u-v) - v^3 + b = 0$$

$$\text{or} \quad u^3 - (u-v)(3uv - a) - v^3 + b = 0$$

If we can find u and v such that

$$3uv = a \quad \text{and} \quad u^3 - v^3 + b = 0 \tag{2}$$

we have a solution. Let $U = u^3$, $V = v^3$

$$\text{Then} \quad UV = \frac{a^3}{27} \quad \text{and} \quad U - V + b = 0$$

$$\therefore U^2 - UV + bU = 0 \quad \text{so that} \quad U^2 + bU - \frac{a^3}{27} = 0$$

It follows that

$$U = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \frac{a^3}{27}}$$

whence

$$V = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \frac{a^3}{27}} \; (= U + b).$$

Taking the positive roots we are lead to write

$$x = u - v = \sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \frac{a^3}{27}}} - \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \frac{a^3}{27}}} \tag{*}$$

We need choose the cube roots in such a way that $uv = \frac{a}{3}$. There are 3 such pairs of cube roots and this then gives 3 roots to the cubic. (There are three possible values of the cube root for u but once this is chosen v is specified uniquely.)

The formulae (*) are now called **Cardan's formulae**. We note:

(i) If a is positive and b is negative (as would now correspond to Del Ferro's equation) we obtain a positive solution. This is the case that Tartaglia and his cohorts could most easily answer. But the formula is not as useful as we might think! Let's consider as an example $x^3 + 3x - 4 = 0$ (or $x^3 + 3x = 4$).

By inspection it is immediate that $x = 1$ is a solution. Cartan's formula yields a root $\sqrt[3]{2+\sqrt{5}} - \sqrt[3]{-2+\sqrt{5}}$. Does this equal 1? The answer is "yes" as a calculation on a calculator suggests. But how do you prove it? Perhaps you can find some trick but straightforward transformations lead to cubic radicals that cannot be removed. (Probably this was the reason Fior could not solve Tartaglia's cubic equation.)

(ii) The situation is even more confusing in the case of three real roots. This is called the irreducible case. In this case $\Delta = 27b^2 + 4a^2 < 0$ (can you prove this?) and the numbers under the cube roots are complex. This is hard to handle even if one has some skill with complex numbers (and remember that in 1545 complex numbers hadn't been invented). If you have some confidence with complex numbers try to solve $x^3 - 6x + 4 = 0$ keeping in mind that $x = 2$ is a root. This is manageable but what about $x^3 - 7x + 6 = 0$ which has roots $-3, 2$, and 1.

In fact Cardano decided to consider not only negative numbers (he called them "purely false") but also complex numbers (which he called "truly sophisticated"). Needless to say he did not properly understand what he was doing. There are two further little points which are worthwhile making. Firstly to use these formulae one must be able to handle complex numbers even if one is interested in real roots only. (This is not the case for quadratic equations.) Secondly there is an alternative method which is excellent in handling cubics which have real roots only. We will explain this approach in a second article.