SOLUTIONS TO THE COMPETITION PROBLEMS

JUNIOR

1. a and b are positive integers. Of the following statements, three are true, one is false.

   (i) $a+1$ is divisible by b.
   (ii) $a = 2b+5$.
   (iii) $a+b$ is divisible by 3.
   (iv) $a+7b$ is a prime number.

   Determine the possible values of a and b.

ANSWER We first show that the false statement must be (iii), since if it were true both (ii) and (iv) would be false, contradicting the data.
To see this, observe that: (iii) is true $\Rightarrow (a+b)-3b$ is divisible by 3
$\Rightarrow a-2b \neq 5 \Rightarrow$ (ii) is false;
and that (iii) is true $\Rightarrow (a+b)+6b$ is divisible by 3
$\Rightarrow a+7b$ is not a prime number $\Rightarrow$ (iv) is false.
So the correct statements are (i), (ii) and (iv). From (i) and (ii) $2b+6$ is divisible by b, whence b is a factor of 6 i.e. $b=1$, or 2, or 3, or 6.
From (ii) and (iv), $(a+7b) = (2b+5)+7b = 9b+5$ is a prime number. This is not true for $b=1$, or 3. Thus the only values of b for which (i), (ii) and (iv) are all true are $b=2$ and $b=6$. The corresponding values of a obtained from (ii) are 9 and 17 respectively. Thus $(a,b) = (9,2)$ or $(17,6)$.

2. If x is a positive whole number, define $d(x)$ to be the highest common factor of all the numbers obtainable by rearranging the digits of x.
   For example, if x is 402, then $d(x) = 6$, the highest common factor of the numbers 402, 420, 240, 204, 42 (resulting from the rearrangment 042) and 24.
   Find the smallest number x such that $d(x) = 45$, if x must have at least two distinct digits, and the first digit of x cannot be zero.

ANSWER Every digit of x must be 0 or 5 since 5 is a factor of every rearrangement. Since 9 is a factor of x, the sum of the digits must be divisible by 9. It follows that the number of digits equal to 5 must be a multiple of 9. The smallest x will have nine 5's and one 0, the 0 being in the most significant position permitted.
Thus $x = 5055555555$.

3. Simplify the sum

$$\frac{1}{(\sqrt{1}+\sqrt{2})(\sqrt[4]{1}+\sqrt[4]{2})} + \frac{1}{(\sqrt{2}+\sqrt{3})(\sqrt[4]{2}+\sqrt[4]{3})} + \cdots + \frac{1}{(\sqrt{255}+\sqrt{256})(\sqrt[4]{255}+\sqrt[4]{256})}$$

(There are 255 terms and the kth term is $\dfrac{1}{(\sqrt{k}+\sqrt{k+1})(\sqrt[4]{k}+\sqrt[4]{k+1})}$).

ANSWER Since $1 = (k+1) - k = (\sqrt{k+1} + \sqrt{k})(\sqrt{k+1} - \sqrt{k})$
$= (\sqrt{k+1} + \sqrt{k})[(\sqrt[4]{k+1} + \sqrt[4]{k})(\sqrt[4]{k+1} - \sqrt[4]{k})]$.

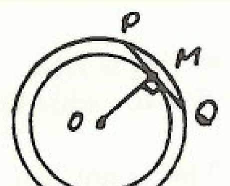
the kth term $= \dfrac{1}{(\sqrt{k} + \sqrt{k+1})(\sqrt[4]{k} + \sqrt[4]{k+1})} = \sqrt[4]{k+1} - \sqrt[4]{k}$.

Thus the sum can be written

$$(\sqrt[4]{2} - \sqrt[4]{1}) + (\sqrt[4]{3} - \sqrt[4]{2}) + \cdots + (\sqrt[4]{k+1} - \sqrt[4]{k}) + \cdots + (\sqrt[4]{256} - \sqrt[4]{255}).$$

Everything cancels out except $\sqrt[4]{256} - \sqrt[4]{1} = 4 - 1$. Hence the sum simplifies to the value 3.

4. The length of the longest line segment that can be fitted into an annulus is 10cm. Calculate the area of the annulus.
(An annulus is the region between two concentric circles).

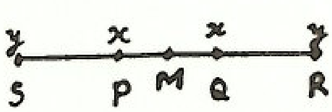


ANSWER It is clear that a longest line segment must have its ends on the circumference of the outer circle, and that it must be tangential to the inner circle. Thus in the figure, PQ is 10cms long, and we have to calculate the area of the annulus. If O is the centre of the circles and M is the point of contact of the tangent, $OM \perp PQ$ and M is the mid-point of PQ. Thus $\overline{MQ} = 5cms$

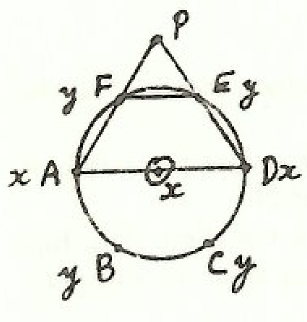
Area of annulus = Area of large circle - Area of small circle
$= \pi \times \overline{OQ}^2 - \pi \times \overline{OM}^2 = \pi(\overline{OQ}^2 - \overline{OM}^2)$
$= \pi \overline{MQ}^2$ (Pythagoras theorem)
$= \pi \times 5^2 = 25\pi$.

5. (i) All the points of a straight line are painted either red or white. Show that we can always find three points A, B, C of the same colour such that the length of AB equals the length of BC.

(ii) All the points of a plane are painted either red or white. Prove that we can always find three points A, B, C in the plane all of the same colour such that $\triangle ABC$ is equilateral.



ANSWER (i) Let the points P and Q both have colour x and let R, S be points on the line such that $\overline{SP} = \overline{PQ} = \overline{QR}$. (See figure). If either R or S has colour x, then that point with P and Q can be taken as A, B and C.
But if R and S both have colour y, then whatever colour is given to M (the mid point both of PQ and of RS) we will have three equally spaced points with the same colour.



(ii) This can be proved by the argument used to establish the more general result in question five on the senior paper (see later), or alternatively as follows:-
Let O and A have colour x. Consider points lying on the circumference of the circle, centre O, through A. Let $ABCDEF$ be a regular hexagon inscribed in this circle.
If B or F has colour x, then ΔAOB or ΔAOF is an equilateral triangle with vertices coloured x and we are finished. So let B and F have colour y.

Then from the ΔBFD we have a y coloured triangle unless D has colour x, which we now assume. Now ΔOCD or ΔODE is an x-coloured triangle unless both C and E have colour y.
Let AF and DE intersect at point P. If P has colour x then ΔAPD is equilateral with all vertices of colour x. If P has colour y, ΔPFE is coloured y. Thus it is impossible to avoid an equilateral triangle with all vertices of the same colour.

6. A machine is programmed to give an integer-valued output $F(N)$ corresponding to any integer input N, subject to the condition

$$F(F(N)) - 3F(N) + 2 = 0$$

(i) If the output is to be some constant K independent of the input, find K.

(ii) Show that, given any natural number M except 1989, it is possible to design the machine so that $F(1989) = M$ and the output is not constant.

ANSWER (i) If $F(N) = K$ for all N, then $F(F(N)) = F(K) = K$. The condition gives $K - 3K + 2 = 0$, whence $K = 1$.

(ii) If $x = F(N)$ for some N then we must have

$$F(x) = F(F(N) = 3F(N) - 2 = 3x - 2 \qquad (*)$$

One way to construct the machine is to program it so that $F(N) = 3N - 2$ if $N \neq 1989$, and $F(1989) = M$.
This is possible only because there is no integer N such that $F(N) = 3N - 2 = 1989$.
For example, if 1989 is replaced by 1990 in the question, one could not have chosen $F(1990)$ arbitrarily. To see this try setting $F(1990) = 664$. Then by (*) $F(664) = 3 \times 664 - 2 = 1990$; and again by (*) $F(1990) = 3 \times 1990 - 2$, in contradiction to $F(1990) = 664$.
[There are many alternative ways to program the machine. For example, let $F(N) = 3N - 2$ if $N = M$ or if $N + 2$ is a multiple of 3, and let $F(N) = M$ otherwise.]

SOLUTIONS TO THE COMPETITION PROBLEMS

SENIOR

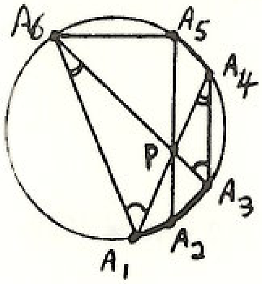
1. If x is a positive whole number, define $d(x)$ to be the highest common factor of all the numbers obtainable by rearranging the digits of x.
For example, if x is 402, then $d(x) = 6$, the highest common factor of the numbers 402, 420, 240, 204, 42 (resulting from the rearrangment 042) and 24.
If x must have at least two distinct digits and cannot begin with the digit zero, find the largest possible value of $d(x)$, and the smallest number x giving that value of $d(x)$. Prove that your answers are correct.

ANSWER Let a, b be two different digits in x. Two of the numbers in the set of arrangements end ....ab and ....ba but are otherwise identical. Since $d(x)$ is a factor of both numbers, it is a factor of their difference, $|(a-b) \times 9|$. Since $|b-a| \leq 9-0 = 9$, we see that for every $x, d(x) \leq 81$, and that $d(x) < 81$ if x contains any digit other than 9 and 0. A number with n 9's and m 0's is divisible by 81 provided the number obtained by replacing the 9's with 1's is divisible by 9, which is true provided n is a multiple of 9. Hence the smallest prossible x, having nine 9's and one 0, is $x = 9099999999$. Since all rearrangments clearly yield numbers still divisible by 81, the maximum possible value of $d(x)$ is achieved by this x.

2. $A_1A_2A_3A_4A_5A_6$ is a hexagon inscribed in a circle, having the property that the diagonals A_1A_4, A_2A_5, and A_3A_6 are concurrent. Prove that

$$\overline{A_1A_2} \cdot \overline{A_3A_4} \cdot \overline{A_5A_6} = \overline{A_2A_3} \cdot \overline{A_4A_5} \cdot \overline{A_6A_1}$$

(where $\overline{PQ}$ denotes the length PQ).



ANSWER Let the point of intersection of the diagonals be P. Note that $A_4\hat{A}_1A_6 = A_4\hat{A}_3A_6$ and $A_1\hat{A}_6A_3 = A_1\hat{A}_4A_3$. $\therefore \Delta PA_1A_6$ is similar to ΔPA_3A_4 whence

$$\frac{\overline{A_1A_6}}{\overline{A_3A_4}} = \frac{\overline{PA_1}}{\overline{PA_3}}$$

Similarly $$\frac{\overline{A_2A_3}}{\overline{A_5A_6}} = \frac{\overline{PA_3}}{\overline{PA_5}} \text{ and } \frac{\overline{A_4A_5}}{\overline{A_1A_2}} = \frac{\overline{PA_5}}{\overline{PA_1}}$$

$$\therefore \frac{\overline{A_1A_6}}{\overline{A_3A_4}} \times \frac{\overline{A_2A_3}}{\overline{A_5A_6}} \times \frac{\overline{A_4A_5}}{\overline{A_1A_2}} = \frac{\overline{PA_1}}{\overline{PA_3}} \times \frac{\overline{PA_3}}{\overline{PA_5}} \times \frac{\overline{PA_5}}{\overline{PA_1}} = 1$$

3. Simplify the sum

$$\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}} + \frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}} + \cdots + \frac{1}{\sqrt[3]{26^2}+\sqrt[3]{26\times 27}+\sqrt[3]{27^2}}$$

where the kth term is

$$\frac{1}{\sqrt[3]{k^2}+\sqrt[3]{k(k+1)}+\sqrt[3]{(k+1)^2}}$$

(i.e. Evaluate $\displaystyle\sum_{k=1}^{26}\frac{1}{\sqrt[3]{k^2}+\sqrt[3]{k(k+1)}+\sqrt[3]{(k+1)^2}}$).

ANSWER Since $\left(\sqrt[3]{k+1}-\sqrt[3]{k}\right)\left((\sqrt[3]{k+1})^2+\sqrt[3]{k+1}\sqrt[3]{k}+(\sqrt[3]{k})^2\right)$
$=(\sqrt[3]{k+1})^3-(\sqrt[3]{k})^3=k+1-k=1$
the kth term in the sum is equal to $\sqrt[3]{k+1}-\sqrt[3]{k}$. The sum is

$$(\sqrt[3]{2}-\sqrt[3]{1})+(\sqrt[3]{3}-\sqrt[3]{2})+(\sqrt[3]{4}-\sqrt[3]{3})+\cdots+(\sqrt[3]{27}-\sqrt[3]{26}).$$

All terms cancel out except for $-\sqrt[3]{1}+\sqrt[3]{27}$.
Hence the sum simplifies to $3-1=2$.

4. The game of "Go" uses a board with 19 equally spaced lines in each of two directions at right angles, together with 181 identical black stones and 180 identical white stones. (The grid of parallel lines divides the board into congruent rectangles, not squares, since the spacing of the lines isn't quite the same for the two directions).
If all 361 stones are placed on the 361 points of intersection, how many different patterns can be so created?
(Two placements will be regarded as the same pattern if a half turn of the board replaces either of them by the other).

ANSWER We shall use $C(n,r)$ to denote the number of different ways to select r elements from a set containing n elements; i.e. $C(n,r)=\frac{n!}{r!(n-r)!}$ where $k!=1\times2\times3\times\cdots\times k$ for $k\in\mathbb{N}$.
The total number of different placements of the stones is $C(361,180)$ since we must select 180 of the 361 intersections to place the white stones. Of these, the ones which are unchanged by a half turn of the board number $C(180,90)$. To see this note that a black stone must be placed on the central intersection. The other 360 intersections fall into 180 pairs each bisected by the central point. Choose 90 of these pairs to contain the white stones.

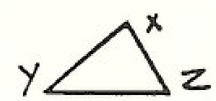
The remaining placements, numbering $C(361,180)-C(180,90)$, yield only $\frac{1}{2}(C(361,180)-C(180,90))$ different patterns (pairing each with the different placement obtained from a half turn of the board).
$\therefore$ The number of different patterns is

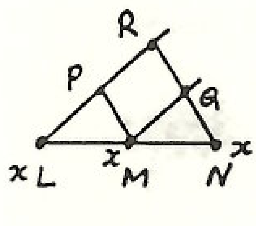
$$\frac{1}{2}(C(361,180)-C(180,90))+C(180,90)=\frac{1}{2}(C(361,180)+C(180,90))$$

5. (i) All the points of a straight line are painted either red or white. Show that we can always find three points A,B,C of the same colour such that the length of AB

equals the length of BC.

(ii) All the points of a plane are painted either red or white. A triangle is given. Prove that we can always find three points A, B, C in the plane, all of the same colour, such that ΔABC is similar to the given triangle.





ANSWER (i) See the answer to question 5 on the Junior paper.
(ii) Let XYZ be the given triangle, placed in the same plane. Choose any line in the plane parallel to YZ, and let L, M, N be equally spaced points on it, all of colour x (using (i)).
Through L and M draw lines parallel to YX, and through M and N draw lines MP and NQR parallel to ZX. (See figure).
Note that P, Q are mid points of LR and NR respectively, so that $PQ \parallel LN$. If any of P, Q, R has colour x, then one of ΔPLM, ΔQMN or ΔRLN is a triangle similar to ΔXYZ, with vertices coloured x. Otherwise, ΔRPQ is similar to ΔXYZ, and has all vertices coloured y.

6. If a, b, c, d are positive integers such that $a + c = 1989$ and $b + d = 1989$, find the largest and smallest possible values of $\frac{a}{b} + \frac{c}{d}$.

ANSWER Without loss of generality, we may suppose $b < d$. Let $x = \frac{a}{b} + \frac{c}{d} = \frac{a(d-b)+1989b}{bd}$ (since $c = 1989 - a$).
$\therefore$ For any b, d (with $b < d$) x is least with $a = 1$ and greatest with $a = 1988$. (*)

Largest possible value of x. From the above, this is $\frac{1988}{b} + \frac{1}{d}$ for some $b, d, d > b$. Since $\frac{1}{d} < \frac{2}{1989} < 1$, if $b \geq 2$ $x \leq \frac{1988}{2} + 1 \leq 995$. But for $b = 1$, $x = 1988 + \frac{1}{1988}$.
Hence the largest possible value of x is $\frac{1988}{1} + \frac{1}{1988}$.

Smallest value of x. Because of (*), take $a = 1$ and let $x(b) = \frac{1}{b} + \frac{1988}{1989-b}$. We wish to find b such that this is as small as possible.

$$x(b) - x(b+1) = \frac{1}{b(b+1)} - \frac{1988}{(1989-b)(1988-b)}$$

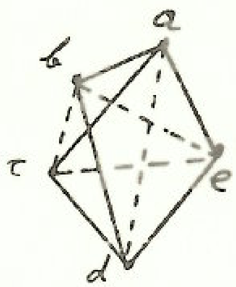
which is greater than zero provided (after some routine algebra)

$$1987b^2 + 5965b - 1988 \times 1989 < 0$$

The positive root of $1987X^2 + 5965X - 1988 \times 1989 = 0$ lies between 43 and 44. The graph is a parabola, concave up. Hence $x(b) - x(b+1) < 0$ if $b \geq 44$ and $x(b) - x(b+1) > 0$ if $b < 44$. Therefore the smallest value of $x(b)$ is $x(44) = \frac{1}{44} + \frac{1988}{1989-44}$

7. Prove that given five irrational numbers, it is always possible to find three of them, x, y, z say, such that $x + y$, $x + z$ and $y + z$ are all irrational numbers.

ANSWER Suppose it is false i.e. suppose we can find five irrational numbers a, b, c, d, e such that for any choice of three of them x, y, z, at least one of $x+y$, $x+z$ and $y+z$ is rational.

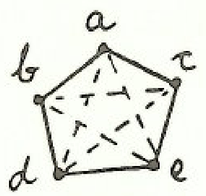


Our argument will be easier to follow if we represent the five numbers by distinct points in a diagram, in which every pair of points is joined either by a red line (if the sum of the two numbers is rational), or by a green line (if the sum is irrational). We are supposing that in this diagram no green "triangle" is formed (i.e. the edges xy, xz, yz are not all green) and of course we want to prove that this is impossible. Certainly there is no red triangle, since if $x+y, y+z$, and $z+x$ are all rational, then $x = \frac{1}{2}[(x+y)+(x+z)-(y+z)]$ would be rational, which is false.

Of the four edges ending at the vertex a, suppose at most one is coloured green. Then there are 3(at least) red ones, ending, say, at u, v, w. To avoid a red triangle auv, uv must be a green line. Similarly vw and uw must be green. But then uvw is a green triangle, contrary to our assumption. Hence at least two edges ending at a must be green.

Similarly, if 3 edges au, av, aw ending at a are green, then to avoid a green triangle all of uv, vw, and uw would have to be red, which we have already observed is impossible.

We conclude that exactly 2 of the edges ending at a must be red, the other 2 green. Of course, the same applies to each of the other vertices.



Let us now rearrange the labels b, c, d, e so that the two red edges ending at a are ab and ac, and the second red edge from b is bd. Now cd cannot be red, since it would be impossible to draw two red edges from e without making a red triangle. The second red edge ending at c (or d) must go to e. Thus we must have $a+b$, $a+c$, $b+d$, $c+e$, and $d+e$ all rational. But this is also impossible, since we would then have

$a = \frac{1}{2}[(a+b)+(a+c)-(c+e)+(e+d)-(b+d)]$, a rational number.

Thus it is impossible to avoid a green triangle, and the required result is established.