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SOLVING GEOMETRIC PROBLEMS WITH ARGUMENTS FROM MECHANICS

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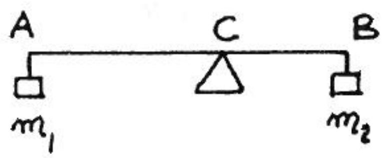
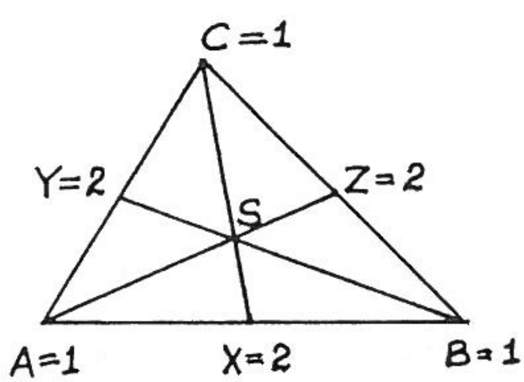
Physics and mathematics have interacted with each other in the course of centuries, each enriching the development of the other. Most problems in physics cannot be solved without a considerable amount of mathematics. On the other hand, sometimes ideas developed in physics or mechanics can be employed to solve problems in mathematics.



One such idea is the centre of mass. Take a light rod AB (meaning that the mass of the rod can be neglected) and suspend two masses, m_1 and m_2 grams, from A and B. Then this rod may be supported at a point C so that it is in equilibrium, where $AC \div CB = m_2 \div m_1$. The whole system behaves as if at C a mass of $m_1 + m_2$ grams would be suspended. Then C is called the centre of mass of this system. Every system consisting of several masses has a unique centre of mass. We are often able to obtain the centre of mass of a system in several different ways, and then the uniqueness will give us a statement in geometry.

Consider ΔABC, with a mass of $1g$ placed at each vertex.

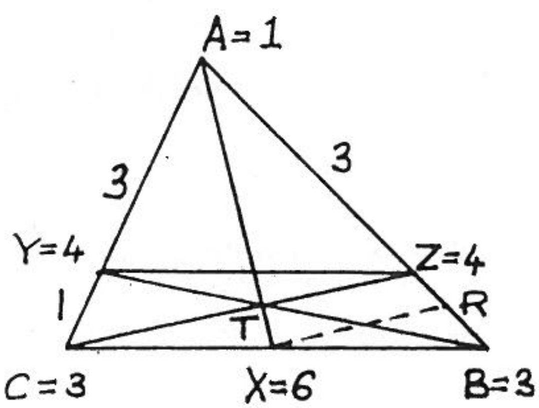


The two masses at A and B may be replaced by a mass of $2g$ at X, the midpoint of AB. Join X to C, then the centre of mass of the whole system must lie on this median CX, and it will be at point S, such that $CS \div SX = 2 \div 1$.

But we could have started with the rod AC, finding that the centre of mass has to lie on YB, where Y is the midpoint of AC. So YB must intersect XC in the point S and $BS \div SY = 2 \div 1$. Similarly AZ passes through S, where Z is the midpoint of BC. This is the (well-known?) theorem that medians of a triangle are concurrent in a point which divides each of them in the ratio $2 \div 1$ (measured from the vertex). Hence this point is called the centroid of the triangle.

Again, let ABC be a triangle, X the midpoint of BC, Y a point on AC, Z a point on AB such that YZ is parallel to BC. Then the lines BY and CZ intersect on AX, the more, if we know the ratio $AY \div YC$, we shall be able to tell in what ratio these 3 lines divide each other.

I prove this theorem in the special case when $AY \div YC = 3 : 1$. (The argument is essentially the same in the case when $AY \div YC = m : n$).



Imagine mass $= 1g$ at A, and masses $3g's$ each at B and C. Then the centre of mass can be found in 3 different ways. X is the centre of mass for B and C, representing $6g's$, so the centre of mass of the whole system is on the line AX, at point T, where $AT \div TX = 6 \div 1$.

Alternately, the centre of mass on AC is at Y with mass 4 so T, the centre of mass of the whole system must also be on YB, and $YT \div TB = 3 \div 4$. Clearly CZ also passes through T, with $CT \div TZ = 4 \div 3$.

We could similarly discuss a case of three general lines drawn from the vertices of a triangle, which are concurrent in a point, and obtain the ratios of the segments on each line. To really appreciate the elegance of this argument, I will prove the above special case with the usual elementary method:

Draw a line through X parallel to CZ, meeting AB in R.

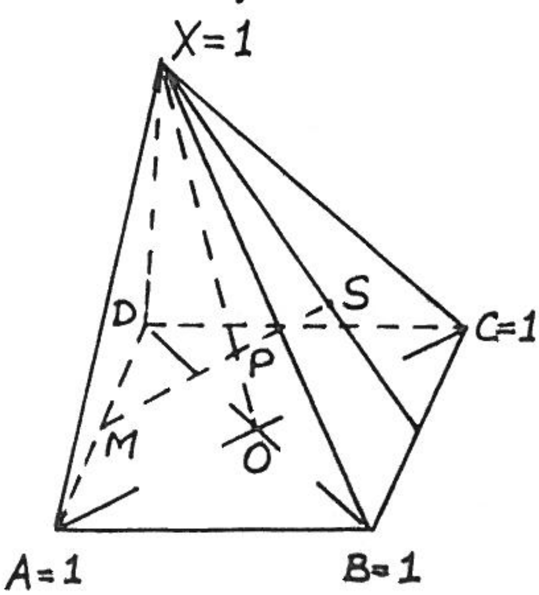
Then

$$\frac{AT}{TX} = \frac{AZ}{ZR} = \frac{AZ}{ZB} \cdot \frac{ZB}{ZR} = \frac{3}{1} \cdot \frac{CB}{CX} = \frac{3}{1} \cdot \frac{2}{1} = \frac{6}{1}.$$

By the symmetry of the figure, BY meets AX in a point T_1 such that $\frac{AT_1}{T_1X} = \frac{6}{1}$, consequently $T_1 = T$ and AX, BY, CZ are concurrent in T. Also ΔYTZ is similar to ΔCTB, so

$$\frac{YT}{TB} = \frac{YZ}{CB} = \frac{YA}{AC} = \frac{3}{4}.$$

The same method can often be used in 3-dimensional space. To illustrate it: Let $ABCD$ represent a square, X a point outside the plane of the square. Then $ABCDX$ is a square pyramid. Imagine a mass of $1g$ at each vertex and let us find the centre of mass in several different ways.



First, the four masses at A, B, C, D can be replaced by a mass of $4g$ at 0, the centre of the square, so the centre of mass will be on $0X$, at point P, where $XP \div P0 = 4 \div 1$. But also: the three masses at B, C, X can be replaced by a mass of 3 at S, the centroid of ΔXBC, and the two remaining masses at A and D can be replaced by a mass of $2g$ at M, the midpoint of AD.

So P is also the point where MS meets $X0$, and $MP \div PS = 3 \div 2$, (Note: Two straight lines in 3-space do not always meet each other, but our argument ensures that P is actually a common point to MS and $X0$.) Clearly we could take any one of the side Δs in the place of XBC, therefore we have obtained the following theorem:

In a square pyramid $ABCDX$ the line joining the vertex X to 0, the midpoint of the square, is concurrent at point P with the 4 lines that join the centroid of each side Δ with the midpoint of the opposite edge. Also, $XP : P0 = 4 : 1$ and each of the other 4 lines are divided in the ratio $3 \div 2$, counted from the midpoint of the respective edge.

Here are two problems to try.

1) Let $ABCD$ be a general quadrilateral. Join the midpoints of opposite sides and also the midpoints of the two diagonals. Show that the 3 lines are concurrent at a point P that bisects each of these line segments.

2)a) In tetrahedron $ABCD$ join each vertex to the centroid of the opposite triangle. Show that the 4 lines we have drawn will all be concurrent at a point P which divides each of them in the ratio $3 \div 1$ (measured from the vertex.)

b) If we join the midpoints of opposite edges (e.g. AB and CD), we get 3 lines which will also pass through the above mentioned point P and each will be bisected by P.

The solutions to these problems are on page 19.

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The following quotes are from mathematicians of contrasting beliefs (a "pagan", a "Christian", and an "agnostic").

"When I trace at my pleasure the windings to and fro of the heavenly bodies, I no longer touch the earth with my feet: I stand in the presence of Zeus himself and take my fill of ambrosia, food of the gods"

Ptolemy (c85-c165)

"Why waste words? Geometry existed before the creation, is co-eternal with the mind of God, is God himself (what exists in God that is not God himself?)"

Johannes Kepler (1571-1630)

"When we know the answers we will then know the mind of God"

Stephen Hawking (b 1942)