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DISCRETE VERSUS CONTINUOUS

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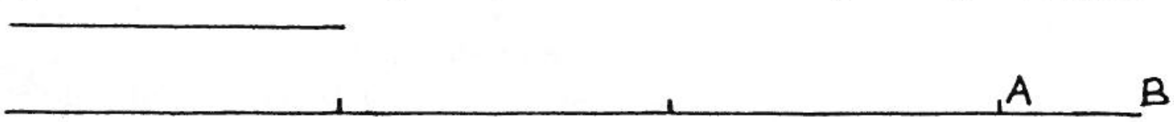
In the beginning, mathematically speaking, there were the whole numbers. A heap of apples can be counted, because it is naturally divided into a fixed number of single apples. But length, time, weight and so on are not "structured *discretely*", like this. Instead of counting, you measure. And to measure, you need to introduce a unit. Since these quantities are *continuous*, or infinitely divisible, the size of the unit is arbitrary – someone just has to decide what should be the length of, say, the standard metre; and then, however small the unit is, there can still be lengths less than the unit, so that the whole numbers are not sufficient for measurement.

The ancient Greeks, tireless thinkers as ever, wondered if it might be possible, in any particular problem, to reduce geometry (of length, time, area and so on) to arithmetic (using only whole numbers), by choosing a unit small enough so that all quantities in the problem would be whole numbers of the unit. To take the simplest example, given two lengths, how do you find a unit, or "common measure", so that both lengths are whole number multiples of it? The Greeks invented an ingenious process for this, called "anthyphairesis": Mark off lengths equal to the smaller length along the larger:

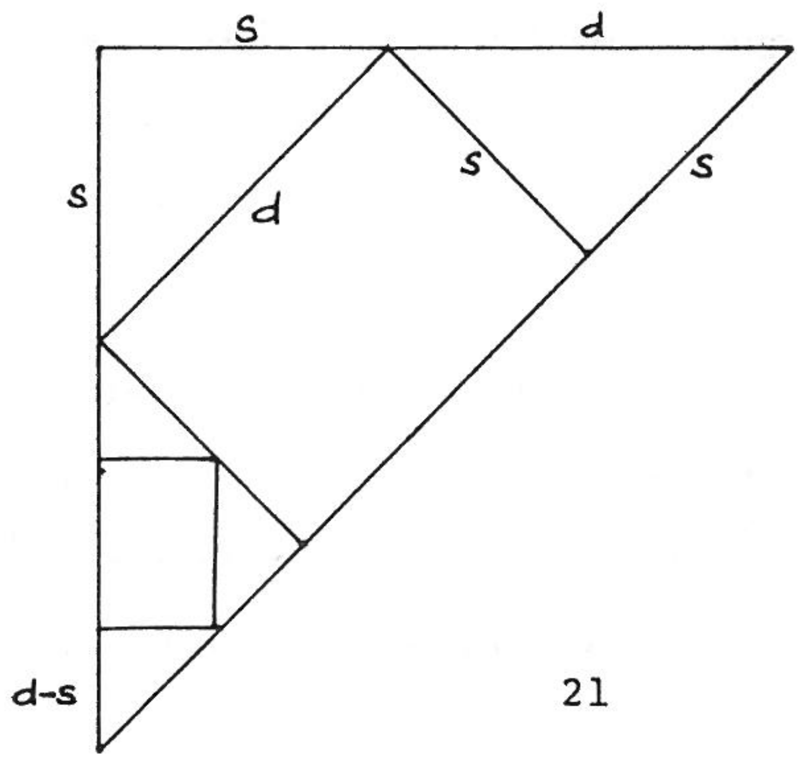


If the smaller length goes exactly into the larger, the smaller is of course itself a common measure. If not, take the small length left over (AB in the picture), and mark off lengths equal to *it* along the smaller of the original two lengths:

If it goes exactly, it is the common measure; if not, repeat the process with the small piece left over at the end, CD (i.e. measure off lengths of CD along AB). Keep doing this until the last length used fits a whole number of times into the one before. The last small length used is the common measure of the original two lengths.

This is very exciting, and extends easily to (for example) three lengths, and to areas. Where the process can be performed, it converts any problem involving geometrical (continuous) quantities into one about whole numbers. No awkward fractions, no decimals, nothing but pure, clean, wholegrain, "natural" numbers. Lovely.

Unfortunately, there is a small bug in the method. Certainly, if there is a unit in terms of which both lengths can be measured, *anthyphairesis* will find it. But there is no guarantee that *anthyphairesis* will ever end – it could be that there are pairs of lengths whose *anthyphairesis* goes on forever. In fact, it was discovered that this happens for the diagonal and the side of any square, so that *the diagonal and the side are incommensurable* (i.e. have no common measure). It is not known how exactly this was discovered, but the following diagram is one possibility:



(Exercise (moderately difficult): Use the above diagram to explain why the *anthyphairesis* of s and d never ends).

It is not known who exactly made the discovery, either. Legend has it that the Pythagorean Brotherhood, devoted to the view that the universe is understandable in terms of (whole) numbers, murdered the first person to reveal it. But this is a much later story, and sounds like those modern myths (that Marilyn Monroe was murdered by the C.I.A., that in the Middle Ages it was believed the earth was flat, and so on), which keep going because they are picturesque, rather than because there is any reason to believe them.

(The incommensurability of the diagonal and the side is logically equivalent to the irrationality of $\sqrt{2}$. We often say loosely, "The Greeks discovered that $\sqrt{2}$ is irrational", but this involves a rather gross reinterpretation of what they really did).

The conclusion drawn was that geometry cannot be reduced to arithmetic. Most of later Greek mathematics therefore concentrated on geometry, as the more general science, and regarded arithmetic as a sub-branch. This meant, for example, that the theory of decimals and fractions took a back seat - possibly to the detriment of the development of mathematics as a useful science. It is a remarkable fact that ancient languages have no unit of speed (like "kilometres per hour").

There has been a certain tension between the continuous and the discrete in mathematics ever since. It is characteristic of medieval thinking that the issue is treated explicitly. Nicole Oresme, the inventor of graphs and probably the best medieval mathematician, presents the story of a dream in which figures representing Arithmetic and Geometry debate as to which is more noble. (The result? – Oresme uses the traditional cop-out of the dreamer waking up at the crucial moment).

In the seventeenth century, (continuous) rates of change came to be understood properly, leading to the development by Newton and Leibniz of the calculus, the science of continuity. The continuous thereafter occupied the forefront of mathematical research, with notable achievements ranging from mathematical physics to the foundations of calculus. Only since about 1950 has the discrete started to regain its place in the sun. This is because computers are "digital" (they work in discrete steps: off and on, 0 and 1, follow this instruction or don't). Like many things to do with computers, even the word "digital" has acquired a kind of gloss, and is used to suggest something is good (as in, "Tonight's prize is this elegantly styled modular digital stereophonic unit, hand-crafted in tooled leatherene by Acme of Australia"). Now, the discrete is everywhere. One of the biggest changes is tertiary mathematics courses in recent times is the introduction of "discrete mathematics" courses, containing mathematics relevant to computing, such as logic and the theory of algorithms and of counting.

Further reading:

D.H. Fowler, 'Ratio in early Greek mathematics', *Bulletin of the American Mathematical Society*, Vol.1 (1979), pages 807-846.

Nicole Oresme and the Kinematics of Circular Motion, edited and translated by E. Grant.

A. Ralston, 'Discrete mathematics: the new mathematics of science', *American Scientist*, Vol.74 (1986), pages 611-618.