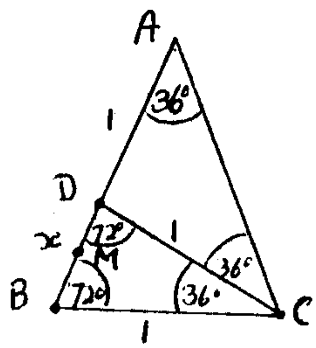
Q.773 Prove that $\tan 36° \times \tan 72° = \sqrt{5}$.

ANSWER Let ΔABC be isosceles with $B\hat{A}C = 36°$ and BC of length 1.



Let CD bisect the angle $A\hat{C}B$. It is easy to calculate the other angles shown on the figure:- $A\hat{B}C = A\hat{C}B = \frac{1}{2}(180° - 36°) = 72°$. Thus ΔBCD and ΔADC are isosceles and the lengths CD and DA are equal to 1.

If x = length BD, since ΔBDC is similar to ΔCBA, $\frac{x}{1} = \frac{DB}{BC} = \frac{CB}{BA} = \frac{1}{1+x}$. Therefore $x^2 + x - 1 = 0$ so that $x = \frac{-1+\sqrt{1+4}}{2} = \frac{-1+\sqrt{5}}{2}$. (We have discarded the negative solution). If M is the midpoint of BD, ΔCMB is right angled at M, so $\cos 72° = \frac{BM}{BC} = \frac{x}{2} = \frac{\sqrt{5}-1}{4}$.

Now

$$\begin{aligned}\tan 36° \times \tan 72° &= \frac{\sin 36°}{\cos 36°} \times \frac{2\sin 36° \cos 36°}{\cos 72°}\\ &= \frac{2\sin^2 36°}{\cos 72°} = \frac{1-\cos 72°}{\cos 72°}\\ &= (1 - \frac{\sqrt{5}-1}{4}) \times \frac{4}{\sqrt{5}-1}\\ &= \frac{5-\sqrt{5}}{\sqrt{5}-1} = \sqrt{5}.\end{aligned}$$

Q.774 Let x be a whole number none of whose digits is 0. Let $d(x)$ be the highest common factor of all the numbers obtainable by rearranging the digits of x.

e.g. if $x = 468$, $d(x) = 18$ since this is the h.c.f. of $\{468, 486, 648, 648, 684, 846, 864\}$.

If the digits of x are not all the same, find the largest possible value of $d(x)$, and the smallest value of x for which $d(x)$ has that maximum value.

ANSWER Let a, b be two different digits in x. Two of the numbers in the set of arrangements end $\cdots ab$ and $\cdots ba$ but are otherwise identical. Since $d(x)$ is a factor of both numbers, it is a factor of their difference, $|(a-b)\times 9|$. Since $|b-a| \leq 9-1 = 8$, we see that for every $x, d(x) \leq 72$, and $d(x) < 72$ if x has any digits other than 1 and 9. But clearly if $d(x) = 72$ (or any even number for that matter) every digit in x must be even. Thus $d(x) = 72$ is not achievable. The next largest possible value of $|b-a|$ is 7, when $\{a, b\} = \{1, 8\}$ or $\{2, 9\}$. If the value $d(x) = 9\times 7 = 63$ is achievable, all digits of x must be 1's and 8's or else 2's and 9's. Since 9 is a factor of x, the sum of the digits must be a multiple of 9. If there are m $2's$ and n $9's$, m must be a multiple of 9 to make $2m + 9n$ a multiple of 9. But $m \times 1 + n \times 8$ is a multiple of 9 whenever $m = n$. Further a number x with m $1's$ and m $8's$ is equal to the number with $2m$ $1's + y$ where y is a number obtained by some arrangement of m $7's$ and m $0's$. (e.g. $1881 = 1111 + 0770$). Hence x (and all its rearrangements) will be divisible by 7 provided the number $\underbrace{11\cdots1}_{2m\text{digits}}$ is divisible by 7. The smallest such multiple of 7 has six digits 111111. Hence 111888, and all its rearrangements, are divisible by both 9 and 7 i.e. by 63. Thus $d(x) = 63$ is achievable, and the smallest x having $d(x) = 63$ is 111888.

Correct solution from P.S. Maylott, (Sydney Grammar School).

Q.775 Let n be any whole number. Find a perfect square with $2n$ digits all less then 7, with the property that if every digit is increased by 3 the resulting number is another perfect square.

[For example, if $n = 1$, 16 is the required number, since 49 is also a square.]

ANSWER Let the smaller square be x^2, the larger y^2.

$$x^2 + \overbrace{33\cdots3}^{2n \text{ digits}} = y^2$$

$$y^2 - x^2 = \frac{10^{2n}-1}{3}$$

$$(y+x)(y-x) = (10^n+1)(\frac{10^n-1}{3}) \quad (1)$$

A possible choice of x, y satisfying (1) is found by solving

$$\begin{cases} y+x = 10^n+1 \\ y-x = \frac{10^n-1}{3} \end{cases}$$

This gives $x = \frac{1}{3}(10^n+2)$, $y = \frac{1}{3}(2\times10^n+1)$. Could we be lucky enough to find that all digits of x^2 are less than 7?

$$x^2 = \frac{1}{9}(10^{2n}+4\times10^n+4) = \frac{1}{9}(10^{2n}-1)+4\times\frac{(10^n-1)}{9}+\frac{1}{9}+\frac{4}{9}+\frac{4}{9}$$

$$= \overbrace{1111\cdots1}^{2n \text{ digits}} + \overbrace{44\cdots4}^{n \text{ digits}} + 1$$

Adding gives a number with $2n$ digits of which the first n are equal to 1, the next $(n-1)$ are equal to 5, and the last is a 6. Thus we have succeeded in finding two suitable squares for any n. For example, if $n = 4$, $3334^2 = 11115556$ and $6667^2 = 44448889$.

Q.776 A list of numbers $a_0, a_1, a_2, \cdots, a_n, \cdots$ has the following properties

(i) $a_0 = 1$

(ii) $a_{n+2} = a_n - 2a_{n+1}$ for all $n \geq 0$.

(iii) a_n remains positive however far the list is extended. (i.e. $a_n > 0$ for all

$n \in \mathbb{N}$)

Show that there is only one such list, and find it.

ANSWER Using (ii) we can calculate $a_2, a_3, \cdots$ in terms of a_1. The first few numbers are

$$1, a_1, 1-2a_1, 5a_1-2, 5-12a_1, 29a_1-12, 29-70a_1, 152a_1-70, \cdots$$

Let c_k denote the kth number in the list

$$\{1, 2, 5, 12, 29, 70, 152, \cdots\}$$

in which $c_k = 2c_{k-1} + c_{k-2}$ for $k \geq 3$.

We see that up to $n = 7$, $a_n = (-1)^n(c_{n-1} - c_n a_1)$.

In fact it is easy to check that this remains true for all n.

[Pf: If $a_k = (-1)^k(c_{k-1} - c_k a_1)$ and $a_{k+1} = (-1)^{k+1}(c_k - c_{k+1}a_1)$, then

$$a_{k+2} = a_k - 2a_{k+1} = (-1)^{k+2}((c_{k-1} + 2c_k) - (c_k + 2c_{k+1})a_1)$$

$$= (-1)^{k+2}(c_{k+1} - c_{k+2}a_1).$$

Hence by mathematical induction the result applies for all n].

Now the condition (iii) translates into

$$(-1)^n(c_{n-1} - c_n a_1) > 0 \text{ for all } n \in \mathbb{N}.$$

i.e. if n is odd, $a_1 > \frac{c_{n-1}}{c_n}$, but if n is even, $a_1 < \frac{c_{n-1}}{c_n}$.

So we must show that there is a unique number a_1 which is less than all of

$$\left\{\frac{c_1}{c_2}, \frac{c_3}{c_4}, \frac{c_5}{c_6}, \quad \ldots, \frac{c_{2k-1}}{c_{2k}} \ldots\right\}$$

but greater than all of

$$\left\{\frac{c_2}{c_3}, \frac{c_4}{c_5}, \frac{c_6}{c_7}, \quad \cdots, \frac{c_{2k}}{c_{2k+1}}, \cdots\right\}$$

Note the following facts about the list $\{c_k\}$:-

a) $c_n \geq 2^n$. [Obvious for $n = 1$ and 2; and if $c_{k-1} \geq 2^{k-1}$ then $c_k \geq 2c_{k-1} \geq 2^k$. Hence true for all n by induction].

b) $c_{n+1}c_{n+2} - c_n c_{n+3} = (-1)^n 2$.

[When $n = 1$ this is true since $2 \times 5 - 1 \times 12 = (-1)^1 \times 2$.

Note that $c_{k+2}c_{k+3} - c_{k+1}c_{k+4}$.

$$= (2c_{k+1} + c_k)c_{k+3} - c_{k+1}(2c_{k+3} + c_{k+2})$$

$$= (-1)[c_{k+1}c_{k+2} - c_k c_{k+3}]$$

Thus if b) holds when $n = k$, it continues to be true when $n = k + 1$. Therefore it is true for all $n \in \mathbb{N}$ by induction].

c) $c_n c_{n+2} - c_{n+1}^2 = (-1)^{n-1}$.

[We omit the proof by mathematical induction, which is very similar to that given for b).]

$$\text{From b)} \qquad \frac{c_{n+2}}{c_{n+3}} - \frac{c_n}{c_{n+1}} = \frac{(-1)^n 2}{c_{n+3}c_{n+2}}. \qquad (*)$$

If n is even, the RHS > 0, and we deduce

$$\frac{c_2}{c_3} < \frac{c_4}{c_5} < \frac{c_6}{c_7} < \cdots < \frac{c_{2k}}{c_{2k+1}} < \cdots \qquad (1)$$

Similarly if n is odd in $(*)$ we deduce

$$\frac{c_1}{c_2} > \frac{c_3}{c_4} > \frac{c_5}{c_6} > \cdots > \frac{c_{2k-1}}{c_{2k}} > \cdots \qquad (2)$$

$$\text{From c)} \qquad \frac{c_n}{c_{n+1}} - \frac{c_{n+1}}{c_{n+2}} = \frac{(-1)^{n-1}}{c_{n+1}c_{n+2}} \tag{3}$$

When n is odd, the RHS in (3) > 0, so for example $\frac{c_5}{c_6} > \frac{c_6}{c_7}$.

In fact from (1), (2) and (3) we can deduce that all fractions in the decreasing list (2) are greater than any of the fractions in the increasing list (1).

(e.g. $\frac{c_{101}}{c_{102}} > \frac{c_{10}}{c_{11}}$ since by (3) $\frac{c_{101}}{c_{102}} > \frac{c_{102}}{c_{103}}$ and by (1) $\frac{c_{102}}{c_{103}} > \frac{c_{10}}{c_{11}}$).

Let α be the smallest number which is greater than every number in the increasing list (1), i.e. such that $\frac{c_{2k}}{c_{2k+1}} < \alpha$ for all k. We have just observed that α is less than every number in the list (2). Hence if β is the largest number less than every number in (2), $\beta \geq \alpha$.

In fact we can show that $\beta = \alpha$. From (3) and a) $\left|\frac{c_n}{c_{n+1}} - \frac{c_{n+1}}{c_{n+2}}\right| < \frac{1}{2^{2n+3}}$, so by taking k large enough we can find a term $\frac{c_{2k}}{c_{2k+1}}$ which differs from $\frac{c_{2k+1}}{c_{2k+2}}$ by as little as desired. Thus it is impossible that $\beta - \alpha > 0$.

We have now proved that a_1 can be chosen in just one way (viz: $a_1 = \alpha = \beta$) to obtain a positive sequence $\{a_n\}$. It remains only to determine the value.

From $c_{k+2} = 2c_{k+1} + c_k$ we have $\dfrac{c_{k+2}}{c_{k+1}} = 2 + \dfrac{c_k}{c_{k+1}}$. Letting $k \to \infty$, $\dfrac{c_k}{c_{k+1}} \to \alpha$ and $\dfrac{c_{k+2}}{c_{k+1}} \to \dfrac{1}{\alpha}$, whence $\dfrac{1}{\alpha} = 2 + \alpha$.

Solving $a^2 + 2\alpha - 1 = 0$ yields $\alpha = -1 + \sqrt{2}$ (since α is not negative).

Thus the only allowable list $\{a_n\}$ is obtained by taking $a_1 = \sqrt{2} - 1$.

Q.777 If m is any given positive integer let N_m be the number of different solutions of

$$w + x + y + z = m$$

where each of the unknowns w, x, y, z denotes a non-negative whole number [e.g. $N_1 = 4$ since there are four solutions of $w + x + y + z = 1$: viz

$$(w, x, y, z) = (1, 0, 0, 0) \text{ or } (0, 1, 0, 0), \text{ or } (0, 0, 1, 0), \text{ or } (0, 0, 0, 1)]$$

Prove that N_m is also the number of different ways of arranging $m1's$ and three 0's in a row, and deduce that $N_m = C(m+3, 3) = \frac{(m+3)(m+2)(m+1)}{6}$.

ANSWER Peter Maylott (Sydney Grammar School) sent the following solution.

Let each solution for $w, x, y, z = m$ produce an arrangement of m $1's$ and three $0's$ in the following manner: $\overbrace{111}^{w1's}0\overbrace{11\cdot 1}^{x1's}0\overbrace{11\cdot 1}^{y1's}\overbrace{11\cdots 1}^{z1's}$ Thus the sum of the $1's$ to the first 0 is equal to w, from the first to the second 0 equal to x, from the second to the third 0 equal to y and from the third 0 to the end equal to z. Thus this gives a combination of m $1's$ and 3 zeros. If any of the unknowns equal zero, then there are no 1's between its designated zeros. By using this method each solution to $w+x+y+z = m$ gives a unique combination of m $1's$ and three zeros and each combination of 1's and zeros gives a unique solution to $w+x+y+z = m$. Therefore N_m equals the number of different ways of arranging m $1's$ and three zeros. To evaluate N_m, consider the arrangements of m $1's$ and three zeros as the number of different ways to select three $1's$ out of $(m+3)1's$ to be replaced by zeros. Using $C(n, r)$, denoting the number of different ways to select r elements out of n elements, we find that

$$\begin{aligned} N_m = C(m+3, 3) &= \frac{(m+3)!}{3!m!} \\ &= \frac{(m+3)(m+2)(m+1)m!}{6 \times m!} \end{aligned}$$

$$\therefore N_m = C(m+3, 3) = \frac{(m+3)(m+2)(m+1)}{6}$$

Q.778 A whole number less than 10000 is chosen randomly. (Zero is a possible choice). What is the probability that the sum of the digits is less than 20?

ANSWER If x is any number < 10000 let $y = 9999 - x$. If the sum of the digits of x is s, the sum of the digits of y is $36 - s$. If n_s is the number of values of x for which the sum is s, it follows from the above observation that $n_s = n_{36-s}$.

Thus $10000 = n_0 + n_1 + \cdots + n_{36}$.

$$= 2n_0 + 2n_1 + \cdots + 2n_{17} + n_{18}$$

from which $n_0 + n_1 + \cdots + n_{17} = 5000 - \frac{1}{2}n_{18}$.

The probability, p, that the sum of digits of x is less than 20 is given by

$$p = \frac{n_0 + n_1 + n_2 + n_{17} + n_{18} + n_{19}}{10000} = \frac{(5000 - \frac{1}{2}n_{18}) + n_{18} + n_{19}}{10000}$$

$$= \frac{1}{2} + \frac{\frac{1}{2}n_{18} + n_{19}}{10000} \quad (1)$$

To finish we need to calculate n_{18} and n_{19}. If the digits of x are $wxyz$, n_{18} is the number of solutions of $w + x + y + z = 18$ (2)

where each symbol is a non negative integer less than 10. By Q.777, the number of <u>all</u> solutions in non-negative integers is $C(18+3,3) = \frac{21\times20\times19}{6} = 1330$. However some of these have $w \geq 10$. How many?

If $w = 10 + W$, $W \geq 0$, $W + x + y + z = 8$. There are $C(8+3,3) = \frac{11.10.3}{6} = 165$ solutions of this.

Similarly 165 of the solutions of (2) have $x \geq 10$, another 165 have $y \geq 10$, and another 165 have $z \geq 10$.

$$\therefore n_{18} = 1330 - 4 \times 165 = 670.$$

An identical argument shows that

$$n_{19} = C(19+3,3) - 4 \times C(9+3,3)$$

$$1540 - 4 \times 220 = 660$$

Finally from (1), the required probability is

$$p = 0.5 + \frac{335+660}{10000} = 0.5995.$$

Q.779 Let a and b be positive integers. When a^2+b^2 is divided by $a+b$ the quotient is q and the remainder is r. Find all pairs (a,b) such that $q^2+r = 1989$.

ANSWER Let us take $a \geq b$. If $a^2+b^2 = q(a+b)+r$ where $0 \leq r < a+b$, then

$q \geq \max\{a-b,\ b\}$ (since $(a+b)(a-b)$ and $(a+b)b$ are each less than a^2+b^2).

If $b \geq \frac{a}{2}$, $r < a+b \leq 3b \leq 3q$.

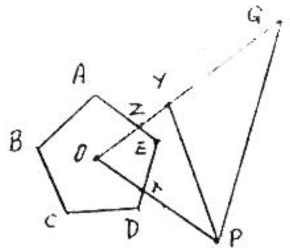
If $b \leq \frac{a}{2}$, $r < a+b \leq \frac{3}{2}a \leq 3(a-b) \leq 3q$.

Thus in every case $r < 3q$. Now $1989 = q^2+r$ if $(q,r) = (44,53)$, or $(43,130)$, or $(42,225)$, etc., but only in the first case is $r < 3q$. Thus $q = 44$, and $r = 53$. We must solve $a^2+b^2 = 44(a+b)+53$. This is equivalent to

$$(a-22)^2+(b-22)^2 = 53+2\times 484 = 1021.$$

The only way to express 1021 as the sum of two perfect squares is $1021 = 900 + 121$ and this yields the solutions $a-22 = 30,\ b-22 = \pm 11$ i.e. $(a,b) = (52,33)$ or $(52,11)$. Of course, the solutions with $b < a$ are $(a,b) = (33,52)$ or (11.52).

Q.780 In the accompanying figure, $ABCDE$ is a regular pentagon whose centre is at O



The following lengths are given: $\overline{DX} = 3cms$, $\overline{XE} = 7cms$; $\overline{OP} = \overline{PY} = \overline{YQ} = 20cms$. Also $\overline{OQ} = \overline{PQ}$. Calculate the area of the quadrilateral $OXEZ$.

ANSWER It is easy to show that $P\hat{O}Q = 72°$ (See a similar triangle in the answer to Q.773.) Since $D\hat{O}E = 72°$ also, $E\hat{O}Z = D\hat{O}X$. Now $\Delta OEZ \equiv \Delta ODX$ since they are equangular, and $OE = OD$.

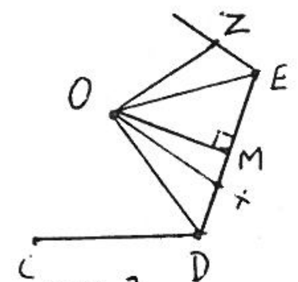


Fig 2

$\therefore$ Area $OXEZ$ = Area $\Delta ODE = \frac{1}{2}DE \times OM$. (Fig. 2)

$$= \tfrac{1}{2}10 \times 5 \times \ cot36° cm^2$$

$$= 34.4cm^2.$$

Q.781 A set S consists of 100 different positive whole numbers, the largest of which is x. The sum of three different numbers chosen from S is never equal to a fourth number in S. Find the smallest possible value of x. Exhibit a set S having this value of x, and prove that no smaller value is possible.

ANSWER The set $\{49, 50, \cdots, 148\}$ consisting of 100 consecutive integers satisfies the requirements. For this set $x = 148$, and we claim that 148 is the smallest possible value of x.

Let m be the smallest element of S. Then $x \geq m + 99 \cdots (1)$

where the equality applies only if S consists of 100 consecutive integers.

The number of essentially different ways of expressing $x - m$ as the sum of two distinct whole numbers is $\frac{(x-m)-1}{2}$ if $(x - m)$ is odd and $(\frac{x-m}{2}) - 1$ if $(x - m)$ is even.

{ Explicitly, $y + z = x - m$ and $y < z$ for $y = 1, 2, \cdots, \frac{x-m-1}{2}, (x - m$ odd) and for $y = 1, 2, \cdots, \frac{x-m}{2} - 1 (x - m \text{ even})\}$.

Since S must not contain both of the numbers y and z with $y + z = x - m$ (except the pair $(y, z) = (m, x - 2m)$) there are at least $\frac{x-m-1}{2} - 1$ positive integers less than x and not in S if $x - m$ is odd, or $\frac{x-m}{2} - 2$ when $(x - m)$ is even.

$\therefore x \geq 100 + \frac{x-m-1}{2} - 1. \cdots (2)$

or $x \geq 100 + \frac{x-m}{2} - 2 \cdots (2')$

These simplify to $x \geq 197 - m \cdots (3)$

or $x \geq 196 - m \cdots (3')$

where (3) applies if $x - m$ is odd, (3′) if $x - m$ is even.

Adding (1) and (3) or (1) and (3′) yields

$2x \geq 296$ or $2x \geq 295$ respectively

$x \geq 148$ or $x \geq 147.5$

Since x is a positive integer it cannot be smaller than 148, as claimed.

(Continued from p.25)

finite number of smaller polygons which can be reassembled, jigsaw fashion, to make a rectangle of dimension $1cm \times Acm$.

(ii) Given any two polygons both of area Acm^2, prove that one can be dissected into a finite set of smaller polygons which can be reassembled to make a figure congruent to the other.

Q.792 There are altogether 120 students at Diophantos High School, the boys outnumbering the girls. To replace breakages in the chemical laboratory, following an unfortunate incident involving a football, each of the boys contributed 30 cents, and each girl 2 cents. The amount collected exactly covered the cost of 12 test tubes at 7 cents each and a number of beakers at 90 cents each.

How many beakers?