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More on Pythagoras and Theano

Michael Deakin[1]

In my previous column, I looked at what we can learn of the mathematical achievements of Pythagoras and Theano. I relied in particular on an article by the Leningrad-based mathematician Leonid Zhmud, although I found his account maddeningly incomplete in places. Here I turn to the things he did *not* say and look at these. In particular, I will discuss the connection that Pythagoras may have had with:

1. the "construction" of the regular polyhedra
2. the discovery of irrationality
3. the properties of the Golden Mean.

Here Zhmud is silent or almost so, and we need to look elsewhere for guidance. The best source for these topics is a paper published in 1945 in the journal *Annals of Mathematics*. The author was Kurt von Fritz of Columbia University in the USA. He titled this work "The discovery of incommensurability by Hippasus of Metapontum". As this title makes clear, the discussion is especially relevant to the second item listed above, but in fact it deals with all three.

First for some preliminaries. Metapontum was the home of the Pythagorean society, and so Hippasus is identified as a Pythagorean. There is a brief but informative article on him in the Wikipedia,

http://en.wikipedia.org/wiki/Hippasus.

This reads in part:

> Pappus merely says that the knowledge of irrational numbers originated in the Pythagorean school, and that the member who first divulged the secret perished by drowning. Iamblichus gives a series of inconsistent reports. In one story he explains how a Pythagorean was merely expelled for divulging the nature of the irrational; but he then cites the legend of the Pythagorean who drowned at sea for making known the construction of the regular dodecahedron in the sphere. In another account he tells how it was Hippasus who drowned at sea for betraying the construction of the dodecahedron and taking credit for this construction himself; but in another story this same punishment is meted out to the Pythagorean who divulged knowledge of the irrational. Iamblichus clearly states that the drowning at sea was a punishment from the gods for impious behaviour.

[1]Dr Michael Deakin is an Adjunct Senior Research Fellow in Mathematics and Statistics at Monash University.

Here in full is what Iamblichus had to say:

> [The Pythagoreans did not] think fit either to speak or write in such a way, that their conceptions might be obvious to any casual persons; but Pythagoras is said to have taught this in the first place to those who came to him, that, being purified from all incontinence, they should preserve in silence the doctrines they had heard. It is said, therefore, that he who first divulged the theory of commensurable and incommensurable quantities, to those who were unworthy to receive it, was so hated by the Pythagoreans that they not only expelled him from their common association, and from living with them, but also constructed a tomb for him, as one who had migrated from the human and passed into another life. Others also say, that the Divine Power was indignant with those who divulged the dogma of Pythagoras: for that he perished in the sea, as an impious person, who rendered manifest the composition of the *icostagonus*; viz. who delivered the method of inscribing in a sphere the dodecæhedron, which is one of what are called the five solid figures. But according to others, this happened to him who unfolded the doctrine of irrational and incommensurable quantities.

So Hippasus was a Pythagorean, who somehow fell out with the brotherhood, and was in some way punished either by his colleagues or else by the gods for some piece of dereliction. Usually, he is seen as a late member of the Pythagorean circle, but as the above quotes make clear, there is not much else that can be agreed.

Possibly he did discover the irrationality (incommensurability) of some number; possibly he broke the rule of silence, and possibly did so in the matter of the "construction" of the regular dodecahedron. But let us look at how these topics interact with one another.

Let us start the story with the *pentagram*. This is a regular 5-pointed star as shown in the figure.

The five lines that make up the figure protrude beyond the boundaries of a regular *pentagon*. Indeed, we could start with this pentagon and extend its sides until they intersect with one another and so produce the pentagram. Furthermore, although I have not shown it in the diagram, we could draw the diagonals of the enclosed pentagon and so produce another pentagram, smaller than the first and "upside down" in relation to it, but otherwise a similar figure. Alternatively, we could join the points of the original pentagram and so produce another regular pentagon, larger than the first and also "upside down" in relation to it but otherwise completely similar. And so on, both inwards and outwards from the basic figure displayed in the diagram.

It is often asserted that the pentagram was the special symbol of the Pythagorean "brotherhood". So I was a little surprised that Zhmud made no mention of it. However, von Fritz does supply evidence for this assertion. It rests on a single classical

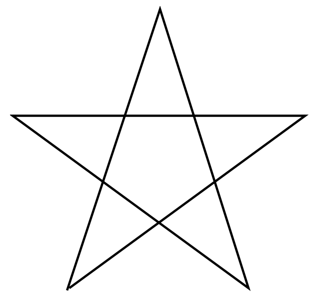

Figure 1: A pentagram.

source, and that a relatively late one, coming from the second century CE. The author Lucian in one of his essays says “Indeed the Pentagram, the triple intersecting triangles, which they [the Pythagoreans] used as a symbol of their sect, they called ‘Health’.”

This is a rather confused passage for the pentagram is *not* made up of three intersecting triangles. (It would be possible to make such a figure, a 9-pointed star, in such a way, just as the Star of David is composed of two intersecting triangles, but this is hardly what Lucian could have had in mind! The word *pentagram* refers unambiguously to the number 5.) However, despite the confusion, it is widely accepted that the pentagram *was* a symbol adopted by the Pythagoreans. Von Fritz follows this path and so shall I.

As indicated above and as illustrated in the diagram, the pentagram is closely associated with the pentagon, and it is from 12 pentagons that the dodecahedron is constructed. Moreover, in the Southern regions of Italy (Metapontum) where the Pythagoreans set up their community, there are naturally occurring dodecahedra. Crystals of pyrite take this form. So it would certainly be a very reasonable thing for the Pythagoreans to set about exploring the properties of the pentagram, the pentagon and the dodecahedron.

This brings us back to Hippasus. One of the stories of him has him “betraying the construction of the dodecahedron”, or in other words, passing on the (secret) knowledge of how this may be done. Zhmud restricts Pythagoras’s accomplishment to the construction of the first two of the platonic solids (presumably the tetrahedron and the cube) and only these, but he gives no reason for this conclusion. When, centuries later, Euclid considered the “construction” of these solids, he meant that he was demonstrating their relation to a sphere passing through all their vertices. What with naturally occurring dodecahedra there before them, the question of their construction would be an obvious one.

It is also apposite to ask about Hippasus’s possible connection to the discovery of the irrationals, as mentioned in a similar context by Iamblichus. I think it fair to say that when we think of the discovery of the irrational, our thoughts immediately home in on $\sqrt{2}$. Certainly, the oldest known proof of irrationality deals with this number. It comes from Aristotle’s *Prior Analytics*(Translated by A. J. Jenkinson available at ebooks@adelaide 2007), and it goes as follows:

> For all who effect an argument *per impossibile* infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; e.g. that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this.

A modern version would proceed as follows. Suppose that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ where $p,\ q$ are natural numbers and furthermore are the smallest pair satisfying this equation, so one or other must be odd (because otherwise a 2 could be cancelled, so reducing their sizes). From the equation, it follows that $p^2 = 2q^2$, and so p^2 is even. But then p itself must be even, for otherwise p^2 would be odd. So it must have been q which was odd. But now $p = 2r$ for some natural number r, and then $q^2 = 2r^2$. But then q^2 is even and this means that q must be even, when we have just shown that it has to be odd! It follows that our original assumption cannot hold and so $\sqrt{2}$ cannot be rational.

A variant of this argument could run somewhat differently but to the same effect. Start with two natural numbers $p,\ q$ for which $p^2 = 2q^2$, and then deduce, as above, that $q^2 = 2r^2$. In the same way, we can show that $r^2 = 2s^2$ and further that $s^2 = 2t^2$, and so on. We set up an infinite sequence of natural numbers $p,\ q,\ r,\ s,\ ...$ each smaller than its predecessor. But, however large a p we begin with, there will necessarily be only a finite number of natural numbers less than p. An infinite sequence such as we deduced is an impossibility.

It is possible to give a geometric version of this argument, and given the fondness of the ancient Greek tradition for geometric reasoning, this is a distinct possibility for the earliest version of the argument for the irrationality of $\sqrt{2}$. Such an argument was in fact sketched out, without full detail, by a much later Greek mathematician, Proclus. It was used as the cover diagram for *Function* (April 1999), and so I won't repeat it here. The full details are actually somewhat complicated.

A variation on this theme was provided last century by the American mathematician Ivan Niven. He supposed, as above, that there exist natural numbers $p,\ q$ such that $q\sqrt{2} = p$ and that q is the smallest integer for which this is possible. Write $q^* = q\sqrt{2} - q$. Clearly q^* is a natural number, because it is the difference of two natural numbers, of which the first is larger than the second. Clearly also $q^* < q$. But $q^*\sqrt{2} = q\left(\sqrt{2}\right)^2 - q\sqrt{2} = 2q - p$, which is a natural number, because $2q > p$. But we had previously made q itself the smallest natural number with this property, so that once again, a contradiction is reached. Alternatively, we could recast this in the "infinite sequence" form, with a succession of denominators $q,\ q^*,\ q^{**},\ ...$, each less than its predecessor.

However, Von Fritz suggested that the number whose irrationality was first proved was not in fact $\sqrt{2}$, but rather another (at first sight less obvious) one: $\frac{1+\sqrt{5}}{2}$. This number is known as the Golden Ratio, commonly denoted by the symbol φ; it has a direct connection to the pentagon and the pentagram. Look again at the diagram. Consider the length of the line (not shown) joining two adjacent points of the pentagram. Call this length a. Call the length of the line joining two points lying *on* the pentagram b. Then it may be proved that $b = \varphi a$. Now take the difference between a, b. This is $(\varphi - 1)\, a$. But $\varphi - 1 = \frac{\sqrt{5}-1}{2} = \frac{1}{1+\sqrt{5}} = \frac{1}{\varphi}$. So $b - a = \frac{a}{\varphi}$, and if we lay off the difference between the two lengths b, a against one another, we produce the same ratio that we had before: $a = \varphi\,(b-a)$. Von Fritz produced a very elegant diagram demonstrating this in terms of the nests of pentagons and pentagrams that can be embedded inside an initial pentagon. This provides an "infinite sequence" demonstration of the irrationality of φ, along the same lines as given by Niven for $\sqrt{2}$. The denominator $b-a$ replaces a, and the process may now be repeated infinitely many times to produce the infinite sequence.

A different (and simpler) diagram was produced by a former colleague of mine, the late Chris Ash. It is given in fuller detail than I provide here in John Crossley's book *The Emergence of Number* (Singapore: World Scientific, 1987). A proof very like this one is one of those given on the Wikipedia website on the Golden Mean.

There is another approach to the question. This uses *continued fractions*. A continued fraction is an expression of the form given below, where the letters all stand for integers.

$$a + \frac{b}{c + \frac{d}{e + \frac{f}{g + \dots}}}$$

The numerators are the numbers b, d, f, ... and the denominators are: c, e, g, ... If one of the numerators is zero, the fraction is said to terminate and is described as *finite*; otherwise it does not terminate and is described as *infinite*. It is obvious that if the fraction terminates, then its value is rational; the converse of this statement is also true, although I won't stop to prove it here: an infinite continued fraction necessarily represents an irrational number. In the event that all the numerators are equal to 1, the continued fraction is said to be a *simple continued fraction*.

An even further simplification is achieved if all the other numbers (that is to say a and all the denominators) are equal. In that case the continued fraction is

$$n + \frac{1}{n + \frac{1}{n + \frac{1}{n + \dots}}} = f(n),$$

say.

Now it is obvious (just by looking at it!) that

$$f(n) = n + \frac{1}{f(n)}$$

and this is a quadratic equation whose (positive) solution is

$$f(n) = \frac{1}{2}\left(n + \sqrt{n^2+4}\right).$$

In particular, if we, set $n = 1$, we would have

$$f(1) = \frac{1}{2}\left(1 + \sqrt{5}\right) = \varphi$$

the Golden Ratio. But we can also look at matters another way, and note that $f(1)$ is the simplest non-teminating continued fraction we could possibly have, and so in a sense, it is the simplest irrational number we could possibly have![2]

So, in a sense it would be *fitting* if φ were the first number whose irrationality was proved. This is not to claim that things really happened this way. They might have and von Fritz has argued that indeed they did. It is unlikely that we will ever know for certain.

And what about Theano? Well, she was clearly an important member of the Pythagorean circle, and so would have been interested in the Golden Ratio. However, she lived before the time of Hippasus, and so may well not have known of φ's irrationality. As to the claim that she wrote a book on φ, well all we can say is that she may have done. It was the sort of thing she might have done, but there is absolutely no evidence that she actually did.

[2]It is interesting to look also at $f(2)$. This has the value $1 + \sqrt{2}$, and is sometimes called the *silver ratio*.