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The Fibonacci Pi Series

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The purpose of this paper is both to observe, understand and appreciate the link between the Fibonacci sequence and the ubiquitous mathematical constant, π. It proves the following series for π, making use of the Fibonacci numbers.

$$\pi = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{4 \cdot (-1)^k}{(2k+1)(F_{2n+1})^{2k+1}} \tag{0.1}$$

where F_n is the nth Fibonacci number. The derivation of this identity employs several different mathematical concepts.

Angle Sum Formulae

We begin with the well-known trigonometric sum formulae:

$$\sin(A+B) = \sin(A) \cdot \cos(B) + \cos(A) \cdot \sin(B) \tag{0.2}$$

$$\cos(A+B) = \cos(A) \cdot \cos(B) - \sin(A) \cdot \sin(B). \tag{0.3}$$

By dividing these expressions, we obtain a corresponding formula for the tangent:

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\tan(A) + \tan(B)}{1 - \tan(A) \cdot \tan(B)} \tag{0.4}$$

Inverse Tangent Addition Formula

To advance further, we now consider the inverse tangent function $\tan^{-1}(x)$. Taking A and B as $\tan^{-1} x$ and $\tan^{-1} y$ respectively, we see that:

$$\tan(\tan^{-1}(x) + \tan^{-1}(y)) = \frac{\tan(\tan^{-1}(x)) + \tan(\tan^{-1}(y))}{1 - \tan(\tan^{-1}(x)) \cdot \tan(\tan^{-1}(y))} = \frac{x+y}{1 - x \cdot y} \tag{0.5}$$

Hence

$$\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1 - x \cdot y}\right) \tag{0.6}$$

To gain another similar result, which is very relevant to our proof, we take the reciprocals of both A and B, which shows:

$$\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{x+y}{x \cdot y - 1}\right) \tag{0.7}$$

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The Fibonacci Sequence

We first define the Fibonacci Numbers, the most common definition being:

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \tag{0.8}$$

Using this definition of the Fibonacci Numbers to advance our proof, we first prove that the following identity holds:

$$F_{2n+1}^2 - F_{2n}^2 = 1 + F_{2n} \cdot F_{2n+1} \tag{0.9}$$

We can show this through induction. Let $P(n)$ denote the following proposition

$$P(n) = F_{2n+1}^2 - F_{2n}^2 - 1 - F_{2n} \cdot F_{2n+1} = 0 \qquad for\ all\ n \geq 0 \tag{0.10}$$

$P(0)$ is clearly true. Now, let k be an integer for which $P(k)$ is true, that is:

$$F_{2k+1}^2 - F_{2k}^2 - 1 - F_{2k} \cdot F_{2k+1} = 0 \tag{0.11}$$

We now consider:

$$F_{2k+3}^2 - F_{2k+2}^2 - 1 - F_{2k+2} \cdot F_{2k+3} \tag{0.12}$$

Using the Fibonacci sequences defining recursion, we can see that:

$$F_{2k+2} = F_{2k+1} + F_{2k} \tag{0.13}$$

and

$$F_{2k+3} = F_{2k+2} + F_{2k+1} = 2F_{2k+1} + F_{2k} \tag{0.14}$$

The previous expression for(12) now becomes:

$$(2F_{2k+1} + F_{2k})^2 - (F_{2k+1} + F_{2k})^2 - 1 - (F_{2k+1} + F_{2k}) \cdot (2F_{2k+1} + F_{2k}) \tag{0.15}$$

After expanding and collecting like terms, we arrive at:

$$F_{2k+1}^2 - F_{2k}^2 - 1 - F_{2k} \cdot F_{2k+1} \tag{0.16}$$

Note that this is the same as the expression for $P(k)$, which is also equal to zero by the inductive hypothesis. As $P(0)$ is true, and $P(k)$ implies $P(k+1)$, we can say that $P(n)$ is true for $n \geq 0$.

Inverse Tangent Addition with Fibonacci Numbers

Returning to the inverse tangent addition formula, we note that:

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{2+3}{2 \cdot 3 - 1}\right) = \tan^{-1}\left(\frac{1}{1}\right) \tag{0.17}$$

$$\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{5+8}{5 \cdot 8 - 1}\right) = \tan^{-1}\left(\frac{1}{3}\right) \tag{0.18}$$

$$\tan^{-1}\left(\frac{1}{13}\right)+\tan^{-1}\left(\frac{1}{21}\right)=\tan^{-1}\left(\frac{13+21}{13\cdot 21-1}\right)=\tan^{-1}\left(\frac{1}{8}\right) \tag{0.19}$$

Thus we can represent $\tan^{-1}(1)$ as a sum of several smaller arctangents, viz,

$$\tan^{-1}(1)=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\tan^{-1}\left(\frac{1}{21}\right) \tag{0.20}$$

Note the denominators correspond to odd-indexed Fibonacci numbers (F_1, F_3, F_5, etc.) truncated by the following even-indexed Fibonacci number. This suggests the generalisation:

$$\tan^{-1}(1)=\tan^{-1}\left(\frac{1}{F_{2m+2}}\right)+\sum_{n=1}^{m}\tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \tag{0.21}$$

We can prove this conjecture by induction. Let $P(m)$ denote the following proposition:

$$P(m)=\frac{\pi}{4}=\tan^{-1}(1)=\tan^{-1}\left(\frac{1}{F_{2m+2}}\right)+\sum_{n=1}^{m}\tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \tag{0.22}$$

for natural numbers $m \geq 1$. We begin with the $P(1)$ case:

$$\begin{aligned}
P(1) &= \tan^{-1}\left(\frac{1}{F_4}\right)+\tan^{-1}\left(\frac{1}{F_3}\right) && (0.23)\\
&= \tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}\left(\frac{2+3}{2\cdot 3-1}\right) && (0.24)\\
&= \tan^{-1}(1)=\frac{\pi}{4} && (0.25)
\end{aligned}$$

Hence, $P(1)$ is true. Now, let k be an integer for which $P(k)$ is true, that is,

$$P(k)=\frac{\pi}{4}=\tan^{-1}(1)=\tan^{-1}\left(\frac{1}{F_{2k+2}}\right)+\sum_{n=1}^{k}\tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \tag{0.26}$$

Now, consider:

$$P(k+1)=\tan^{-1}\left(\frac{1}{F_{2k+4}}\right)+\sum_{n=1}^{k+1}\tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \tag{0.27}$$

Expanding and rearranging, this can be expressed as:

$$\begin{aligned}
P(k+1) &= \tan^{-1}\left(\frac{1}{F_{2k+2}}\right)+\tan^{-1}\left(\frac{1}{F_{2k+3}}\right)\\
&\quad +\tan^{-1}\left(\frac{1}{F_{2k+4}}\right)-\tan^{-1}\left(\frac{1}{F_{2k+2}}\right)\\
&\quad +\sum_{n=1}^{k}\tan^{-1}\left(\frac{1}{F_{2n+1}}\right) && (0.28)
\end{aligned}$$

Using the inductive hypothesis, the first two parts of this evaluate to $\frac{\pi}{4}$, becoming:

$$\frac{\pi}{4} + \tan^{-1}\left(\frac{1}{F_{2k+3}}\right) + \tan^{-1}\left(\frac{1}{F_{2k+4}}\right) - \tan^{-1}\left(\frac{1}{F_{2k+2}}\right) \tag{0.29}$$

As we wish to show that the entirety of this expression evaluates to $\frac{\pi}{4}$, we need to show that the remaining part of the expression evaluates to 0, that is,

$$\tan^{-1}\left(\frac{1}{F_{2k+3}}\right) + \tan^{-1}\left(\frac{1}{F_{2k+4}}\right) - \tan^{-1}\left(\frac{1}{F_{2k+2}}\right) = 0 \tag{0.30}$$

This can also be stated as:

$$\tan^{-1}\left(\frac{1}{F_{2k+3}}\right) + \tan^{-1}\left(\frac{1}{F_{2k+4}}\right) = \tan^{-1}\left(\frac{1}{F_{2k+2}}\right) \tag{0.31}$$

By shifting the index from $k+1$ to k, this can be stated simply in terms of F_{2k} and F_{2k+1} as:

$$\tan^{-1}\left(\frac{1}{F_{2k+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2k} + F_{2k+1}}\right) = \tan^{-1}\left(\frac{1}{F_{2k}}\right) \tag{0.32}$$

Using our previous arctangent addition formula, we condense this to:

$$\tan^{-1}\left(\frac{F_{2k} + 2F_{2k+1}}{F_{2k}F_{2k+1} + F_{2k+1}^2 - 1}\right) = \tan^{-1}\left(\frac{1}{F_{2k}}\right) \tag{0.33}$$

As both sides of the equation are expressed in terms of the inverse tangent, we take the tangent of both sides and rearrange slightly to arrive at:

$$F_{2k}^2 + 2F_{2k}F_{2k+1} = F_{2k}F_{2k+1} + F_{2k+1}^2 - 1 \tag{0.34}$$

Rearranging further:

$$F_{2k+1}^2 - F_{2k}^2 - 1 - F_{2k}F_{2k+1} = 0 \tag{0.35}$$

As we have already proven that this expression is equal to zero, the inductive proof is complete, and we can say that

$$\frac{\pi}{4} = \tan^{-1}(1) = \tan^{-1}\left(\frac{1}{F_{2m+2}}\right) + \sum_{n=1}^{m} \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \tag{0.36}$$

for any natural number $m/ge1$. In addition, since $F_m \to \infty$ as $m \to \infty$, we can say that $\frac{1}{F_m} \to 0$, and similarly, $\frac{1}{F_{2m+2}} \to 0$. Thus, taking a limit as $m \to \infty$, we can omit the extra term, and express it as an infinite series, such that:

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \tag{0.37}$$

Power Series
Our next step is to express the inverse tangent as a power series. Many functions $f(x)$ can be expressed as an infinite power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{0.38}$$

where a_n are a set of coefficients determined by the behaviour of $f(x)$. The simplest such series arises from the geometric series.

Consider:

$$f(x) = 1 + x + x^2 + x^3 + \dots \tag{0.39}$$

It is well-established that $f(x) = \frac{1}{1-x}$. The ratio test shows that this series converges to the function only within the domain $-1 < x < 1$. This can be summarised succinctly as:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \; for \; |x| < 1 \tag{0.40}$$

Replacing x with $-x^2$, we can see that:

$$\sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} \tag{0.41}$$

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \tag{0.42}$$

The Inverse Tangent Power Series
It is well known that power series may be integrated term by term on their interval of convergence. Applying this here, we see that:

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \int \frac{dx}{1+x^2} \tag{0.43}$$

$$\sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \tan^{-1}(x) + c \tag{0.44}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \tan^{-1}(x) \tag{0.45}$$

(checking that $c = 0$ by substituting $x = 0$).

A simple application of the ratio test shows that this power series is only valid on the interval $-1 < x < 1$. It is known that this infinite series also converges to the function $f(x) = \tan^{-1}(x)$ on the boundary of its radius of convergence, that is, at $x = -1$ and $x = 1$.

Having arrived at an infinite series expansion for $f(x) = \tan^{-1}(x)$, it is tempting to simply substitute in $x = 1$ and multiply by 4 to approximate a value for π [2]. However, the problem with this is the rate of convergence of the series.

Consider:

$$\sum_{n=0}^{10} \frac{4 \cdot (-1)^n}{2n+1} \tag{0.46}$$

Taking ten terms of this series returns a π value of 3.232, having no decimal places of accuracy to the actual value of π $(3.142\ldots)$. However, when x is a much smaller number, the approximation to the arctangent function converges much faster, as the terms are similarly smaller. For this reason, we prefer to express π in terms of the arctangent of smaller ratios, greatly accelerating the rate of convergence.

Final Proof

Let us now put together the three key facts we have established

$$\frac{\pi}{4} = \tan^{-1}(1) \tag{0.47}$$

$$\tan^{-1}(1) = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \tag{0.48}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \tan^{-1}(x) \tag{0.49}$$

From (46) and (47),

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \tag{0.50}$$

From (48),

$$\tan^{-1}\left(\frac{1}{F_{2n+1}}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) \cdot (F_{2n+1})^{2k+1}} \tag{0.51}$$

Substituting (50) into (49), we have:

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) \cdot (F_{2n+1})^{2k+1}} \tag{0.52}$$

Transposing the factor of 4 across to the other side of the equation, we arrive at

$$\pi = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{4(-1)^k}{(2k+1) \cdot (F_{2n+1})^{2k+1}} \tag{0.53}$$

This is the desired identity - the Fibonacci Pi Series.

Convergence Rates

[2] This method is known variously as the Leibniz, Gregory or Madhava series method.

We must consider its rate of convergence to π relative to alternate methods. Defining

$$\pi_{a,b} = \sum_{n=1}^{a} \sum_{k=0}^{b} \frac{4(-1)^k}{(2k+1) \cdot (F_{2n+1})^{2k+1}} \tag{0.54}$$

We now construct a table of values for some different values of a and b.

a	b	$\pi_{a,b}$	Error=$\lvert\pi_{a,b} - \pi\rvert$
1	1	1.83333333333	$\approx 10^0$
5	5	3.11378443244	$\approx 10^{-2}$
10	10	3.14136682211	$\approx 10^{-4}$
10	25	3.14136680525	$\approx 10^{-4}$
25	10	3.14159267033	$\approx 10^{-8}$
50	20	3.14159265359	$\ll 10^{-8}$

The table demonstrates that large values of a are much more efficient than large values of b in the accurate computation of π.

This is because taking higher values of b only results in more accurate computation of the individual arctangent values, which converge more quickly than the sum as a whole. By contrast, the value of a allows for the arctangent terms to be summed much more rapidly.

Those seeking to calculate π using a truncated form of our infinite Fibonacci Pi Series should use the largest possible value of a for their approximation and relatively smaller values of b.