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2010 University of New South Wales School Mathematics Competition [1]

Junior Division – Problems and Solutions

Problem 1

Find the set of all pairs of positive integers (n, m) that satisfy

$$\left|n^2 - m^2 - 2010\right| \leq 1.$$

Solution 1

We being by factoring then we seek n and m that satisfy one of the following:
(i) $(n-m)(n+m) = 2009 = (1)(7)(7)(41)(2009)$ or
(ii) $(n-m)(n+m) = 2010 = (1)(2)(3)(5)(67)(2010)$ or
(iii) $(n-m)(n+m) = 2011 = (1)(2011)$.
From (i) we have the three possibilities $n-m = (1), n+m = (2009)$, $n-m = (41), n+m = (7)(7)$, $n-m = (7), n+m = (7)(41)$, with respective solutions $(n,m) = (1005, 1004)$, $(n,m) = (45,4)$,$(n,m) = (147,140)$. There are no integer solutions for (ii) since the factorisation has either $(n-m)$ even and $(n+m)$ odd, or vice versa, and in case (iii) we have $n-m = (1), n+m = (2011)$ with solution $(n,m) = (1006, 1005)$. The set of all pairs of positive integers that satisfy $|n^2 - m^2 - 2010| \leq 1$ is

$$\{(1005, 1004),\ (45, 4),\ (147, 140),\ (1006, 1005)\}.$$

Problem 2

Show that if n is a positive integer then $n(n+1)(n+2)(n+3)+1$ is a perfect square and deduce that the product of four consecutive positive integers is never a perfect square.

Solution 2

[1]The problems and solutions were compiled, created, refined with contributions from David Angell, Peter Brown, David Crocker, Ian Doust, Bruce Henry (Director), Mike Hirschhorn, David Hunt and Thanh Tran.

$$
\begin{aligned}
x(x+1)(x+2)(x+3)+1 &= [(x+1)(x+2)][(x(x+3)]+1 \\
&= (x^2+3x+2)(x^2+3x)+1 \\
&= [(x+\frac{3}{2})^2-\frac{1}{4}][(x+\frac{3}{2})^2-\frac{9}{4}]+1 \\
&= (x+\frac{3}{2})^4-\frac{5}{2}(x+\frac{3}{2})^2+\frac{25}{16} \\
&= ((x+\frac{3}{2})^2-\frac{5}{4})^2 \\
&= (x^2+3x+1)^2
\end{aligned}
$$

Alternate solutions to this part are

$$
\begin{aligned}
x(x+1)(x+2)(x+3)+1 &= x^4+6x^3+11x^2+6x+1 \\
&= (x^4+4x^3+6x^2+4x+1)+2x^3+5x^2+2x \\
&= (x+1)^4+2x(x^2+2x+1)+x^2 \\
&= (x+1)^4+2x(x+1)^2+x^2 \\
&= ((x+1)^2+x)^2 \\
&= (x^2+3x+1)^2
\end{aligned}
$$

and

$$
\begin{aligned}
x(x+1)(x+2)(x+3)+1 &= (x(x+3))((x+1)(x+2))+1 \\
&= (x^2+3x)(x^2+3x+2)+1 \\
&= (x^2+3x)^2+2(x^2+3x)+1 \\
&= ((x^2+3x)+1)^2 \\
&= (x^2+3x+1)^2
\end{aligned}
$$

Suppose that the product of four consecutive integers

$$n, n+1, n+2, n+3$$

is a perfect square, that is,

$$n(n+1)(n+2)(n+3) = a^2, \quad a \in \mathbf{N}.$$

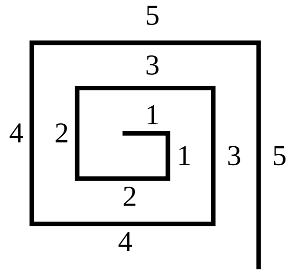
We also have

$$n(n+1)(n+2)(n+3)+1 = b^2, \quad b \in \mathbf{N}.$$

Thus we require $1 = b^2 - a^2 = (b-a)(b+a)$ and $b-a, b+a \in \mathbf{Z}^+$ so $b-a = b+a = 1$ and hence the only possibility has $a = 0$ and $b = 1$. Thus there are no positive integers n to satisfy $n(n+1)(n+2)(n+3) = 0$.

Problem 3

A monk sets out from a monastery in a valley at dawn and follows a winding path up a mountainside at a constant speed, planning to arrive at a temple on the mountaintop at dusk. A second monk sets out from the temple at dawn and travels down the mountainside along the same path, but at twice the speed, until she meets the monk coming up and then they stop for a break together. The temple is at an elevation 945 metres above the elevation of the monastery. When viewed from above the winding path appears as a regular rectangular spiral with the geometry of the central portion as shown below.



The two shortest segments of this spiral have length of 1 metre each and the two longest segments have length 99 metres each. When viewed from the side each straight line segment is at the same constant angle of inclination.

1. What is the change in elevation for each of the monks when they meet?
2. How far has each of the monks travelled when they meet?
3. What is the length of the spiral arm segment on which they meet?

Solution 3

1. Since the angle of inclination is constant the descending monk loses height twice as fast as the ascending monk gains height. Thus the descending monk loses height of 630 metres while the ascending monk gains height of 315 metres.
2. The total length of the spiral segments, when viewed from above, is

$$\begin{aligned} \ell &= 2(1 + 2 + 99 \ldots) \\ &= 2\frac{1}{2}(99)(100) \\ &= 9900 \quad \text{metres} \end{aligned}$$

   The descending monk travels two-thirds of this path, 6600 metres when viewed from above. The total length of the path is from Pythagoras' Theorem

$$\sqrt{9900^2 + 945^2} = 9945 \quad \text{metres.}$$

   Two-thirds of this distance, 6630 metres, is travelled by the descending monk and one-third of the distance, 3315 metres, is travelled by the ascending monk.

3. Suppose that the two monks meet on a spiral arm segment of length k. Then we require either i) $2(1+2+k-1\ldots)+k<6600$ and $2(1+2+k\ldots)>6600$ or ii) $2(1+2+k\ldots)<6600$ and $2(1+2+k\ldots)+(k+1)>6600$ for some integer k. Note that $2(1+2+80\ldots)=6480, 2(1+2+80\ldots)+81=6561, 2(1+2+81\ldots)=6642$ so that condition i) is satisfied for $k=81$. The two monks meet on a spiral arm of length 81 metres, when viewed from above.

An alternate method of solution using trigonometry results in a simpler solution for part (iii). First let α denote the constant angle of elevation of each path segment then the total length of the path is

$$\ell=\sum_{n=1}^{99}2n\sec\alpha=\frac{2.99.100}{2}\sec\alpha=9900\sec\alpha$$

and the height is

$$h=945=\sum_{n=1}^{99}2n\tan\alpha=9900\tan\alpha.$$

From the height we deduce

$$\tan\alpha=\frac{945}{9900}$$

and then

$$\sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+\left(\frac{945}{9900}\right)^2}$$

so that

$$\ell=9900\sqrt{1+\left(\frac{945}{9900}\right)^2}=\sqrt{9900^2+945^2}=9945m.$$

So the first monk travels $\frac{\ell}{3}=3315m$ and the second monk travels $\frac{2\ell}{3}=6630$.

Calculating the distance travelled by the second monk going down we want to find $N\in\mathbf{Z}^+$ such that

$$\sum_{n=1}^{N}2n\sec\alpha\le 6630<\sum_{n=1}^{N+1}2n\sec\alpha.$$

Thus we require

$$N(N+1)\le 6630\cos\alpha=6600<(N+1)(N+2)$$

and since $N(N+1)\approx N^2$ and $\sqrt{6600}\approx 80$ check around $N=80$. Note

$$80.81=6480<6600<81.82=6642$$

then $N=80$ and as $6600-6480=120=81+39$, they meet on a path of base length 81m (the second one for the monk coming down, the first one for the monk going up).

Problem 4

A cubic block can be partitioned into smaller cubic blocks in many ways. An integer n is called a cute-cube number if a cubic block can be partitioned into n cubic blocks of at most two different sizes.

1. Provide an example of a cute-cube number that is greater than 2^3 but less than 3^3.
2. Show that 2010 is a cute-cube number.

Solution 4

1. Fifteen is a cute-cube number. Take a cubic block and partition it into eight cubic blocks of equal size. Take one of these smaller cubic blocks and partition it into eight cubic blocks of equal size.
2. Consider the general construction as follows: Take a cubic block and partition it into n^3 cubic blocks of equal size. Take m of these smaller cubic blocks and partition them into k^3 cubic blocks of equal size then

   $$N = n^3 - m + m(k^3) = n^3 + m(k^3 - 1)$$

   is a cute-cube number.

   Thus $N = 2010$ is a cute-cube number since

   $$\begin{aligned} 2010 &= 6^3 + 69(3^3 - 1) \\ 2010 &= 8^3 + 214(2^3 - 1) \\ 2010 &= 9^3 + 183(2^3 - 1) \\ 2010 &= 11^3 + 97(2^3 - 1). \end{aligned}$$

There are other constructions of cute-cube numbers.

Problem 5

Let N_0 denote a three-digit natural number with not all digits identical. Arrange the digits in descending order and subtract from this number the number that is obtained by arranging the digits in ascending order. Let N_1 denote the result, written as a three-digit number (e.g. 42 is written as 042). Now perform the same operation on N_1 that you performed on N_0 and let N_2 denote the result. Repeat to construct the sequence $N_3, N_4, \ldots$.

1. Show that there exists a number $N^\star$ such that if $N_0 = N^\star$ then $N_1 = N^\star$.
2. Show that $N_6 = N^\star$ for any initial number N_0.

Solution 5

1. Let $a_1 \geq a_2 \geq a_3$ denote the digits of N_0 with $a_1 > a_3$. The digits of N_1 are $10 + a_3 - a_1, 9, a_1 - a_3 - 1$. Thus if N_1 and N_0 have the same digits then the largest of these is $a_1 = 9$. Now equating the remaining two digits we require $\{a_2, a_3\} = \{a_3 + 1, 8 - a_3\}$ which has the solution $a_3 = 4, a_2 = 5$. Thus the digits of N_0 are 9,5,4 and then $N_1 = 954 - 459 = 495,$ so that $N^\star = 495$.

2. More generally, starting with N_0 we construct N_1 with digits $b_1 = 9 \geq b_2 \geq b_3$ and then the digits of N_2 are $10+b_3-b_1, 9, b_1-b_3-1 = b_3+1, 9, 8-b_3$. The largest digit is 9 and the sum of the remaining digits is 9. Note too that the smallest digit is increased by one and thus the larger is decreased by one. This can only continue until the input is 495 which must occur after at most six steps.

Problem 6

Let $\tau(n)$ denote the number of positive factors of a positive integer n. Prove that, for any positive integers m and n, $\tau(mn) \leq \tau(m)\tau(n)$.

Solution 6

Let π, where $\quad i = 1, 2, 3, \ldots$, denote the prime factors of m and the prime factors of n (some of which may be common). Using the product symbol $\prod_{i=1}^{K} x_i$ to denote the product $x_1 \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_K$ we can write

$$
\begin{aligned}
m &= \prod_{i=1}^{K} p_i^{\alpha_i} \quad \alpha_i \geq 0 \\
n &= \prod_{i=1}^{K} p_i^{\beta_i} \quad \beta_i \geq 0 \\
mn &= \prod_{i=1}^{K} p_i^{\alpha_i+\beta_i}
\end{aligned}
$$

The divisors of $p_i^{\delta_i}$ are $1, p_i, \ldots, p_i^{\delta_i}$ so that $\tau(p_i^{\delta_i}) = \delta_i + 1$ and

$$
\begin{aligned}
\tau(m) &= \prod_{i=1}^{K} (\alpha_i + 1) \\
\tau(n) &= \prod_{i=1}^{K} (\beta_i + 1) \\
\tau(mn) &= \prod_{i=1}^{K} (\alpha_i + \beta_i + 1).
\end{aligned}
$$

The result $\tau(mn) < \tau(m)\tau(n)$ now follows since

$$(\alpha_i + \beta_i + 1) \leq (\alpha_i\beta_i + \alpha_i + \beta_i + 1) = (\alpha_i + 1)(\beta_i + 1).$$

An alternate proof is possible without using unique factorisation into primes. This alternate proof starts with the proposition that if d is a divisor of m (i.e. $d|mn$) then $d = d_1 d_2$, where $d_1|m$ and $d_2|n$. It then follows that

$$
\begin{aligned}
\tau(mn) &= |\{d \in \mathbf{Z}^+ : d|mn\}| \\
&= |\{d_1 d_2 : d_1, d_2 \in \mathbf{Z}^+, d_1|m,\ d_2|n\}| \\
&\leq \tau(m)\tau(n).
\end{aligned}
$$

Of course it remains to prove the proposition (see the problems section in this issue).

Senior Division – Problems and Solutions

Problem 1
See Problem 6 in the Junior Competition.

Solution 1
See Problem 6 solution in the Junior Competition.

Problem 2
The infinite order tower power of x is defined as

$$T(x) = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} = x^{(x^{(x^{(x^{\cdot^{\cdot^{\cdot}}})})})}.$$

1. Find the largest number x for which $T(x)$ is finite.
2. Find the value of $T(x)$ in this case.

Solution 2
First we may note that $T(1) = 1$ and thus we seek $x_{max} \geq 1$. Let

$$T(x) = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$$

then take logarithms of each side to obtain

$$\log T = \log x^T = T \log x.$$

Now solve for

$$x(T) = \exp\left(\frac{\log T}{T}\right).$$

To find the maximum of $x(T)$ differentiate x with respect to T then

$$\frac{dx}{dT} = \left(\frac{1 - \log T}{T^2}\right) \exp\left(\frac{\log T}{T}\right).$$

Now $\frac{dx}{dT} = 0$ if $\log T = 1$ which yields $T = e$ and

$$x = \exp(\frac{1}{e}) = 1.444667861\ldots.$$

The above solution assumes that $T(x)$ is a well-defined single-valued function with an inverse $x(T)$ but the situation is more complicated than this, as is apparent by considering $T(\sqrt{2}) = \sqrt{2}^{T(\sqrt{2})}$ which has two solutions $T(\sqrt{2}) = 2$ or $T(\sqrt{2}) = 4$.

If we let $t = T(x)$ and consider $x > 0$ then simple curve sketching arguments reveal that $x^t = t$ has

(i) a unique solution in t for $0 < x \leq 1$

(ii) two solutions in t for $1 < x < e^{\frac{1}{e}}$

(iii) one solution in t for $x = e^{\frac{1}{e}}$

(iv) no solutions in t for $x > e^{\frac{1}{e}}$.

Then taking $x = e^{\frac{1}{e}}$ we have $e^{\frac{t}{e}} = t$ if $t = e$.

Problem 3

A gaoler enters a room with three prisoners and places ten hats in clear view on a table in front of the prisoners. Some of the hats are black and the others are white. The gaoler blindfolds the prisoners and then puts a hat on each of them and removes the remaining seven hats and says, "I will give you turns to deduce the colour of the hat that I have put on your head. If you can do this correctly you will be set free."

He then removes the blindfold from the first prisoner who says, "I can see the colours of the hats of my two fellow prisoners but I cannot tell the colour of my own hat." The gaoler removes the blindfold from the second prisoner who says, "I can see the colours of the hats of my two fellow prisoners but I cannot tell the colour of my own hat." The gaoler is about to remove the blindfold from the third prisoner when the prisoner says, "I cannot tell the colours of the hats of my two fellow prisoners but the colour of my hat is white." The gaoler says, "That's correct, you are now free."

How many black hats did the gaoler bring into the room?

Solution 3

2 black hats.

If there were no black hats then all prisoners would have known the colours of their hats immediately so consider one black hat. It cannot have been on Prisoner 2 or 3 since then Prisoner 1 would have known that he had white. Then Prisoner 2 would know he must have a white hat but he didn't know so there can't have been just one black hat.

Suppose there are two black hats then there are the following possibilities:

P1	P2	P3	P3's Thinking
W	W	W	
W	B	B	Eliminated by statement of Prisoner 1. Obvious
B	W	B	Eliminated by statement of Prisoner 2. Obvious
W	W	B	Eliminated by statement of P2. P2 and P3 cannot both be black as above, so if P3 is black then P2 must be white, but P2 did not know so rule out.
B	W	W	
W	B	W	
B	B	W	

The remaining possibilities all have a white hat on Prisoner 3.

If there were three, four, five, six or seven black hats then in addition to the above there would be the possibility of a black hat on each prisoner in which case Prisoner 3 could not deduce if he had black or white.

If there were eight black hats then there were only two white hats and by swapping black and white in the above table Prisoner 3 would have been led to the conclusion that he had a black hat but this was not his conclusion.

If there was only one white hat then similar to the case of one black hat, Prisoner 3 would have been able to deduce the colours of the hats of all three.

The case of no white hats is trivial.

Problem 4

Consider a triangle with sides a, b, c of unequal length, $a < b < c$. Construct a sequence of triangles $T_1, T_2, ...$ as follows:

Let $s_1 = \dfrac{a+c}{2}$ and let T_1 have sides s_1, s_1, b.

Let $s_2 = \dfrac{s_1+b}{2}$ and let T_2 have sides s_2, s_2, s_1.

Let $s_3 = \dfrac{s_2+s_1}{2}$ and let T_3 have sides s_3, s_3, s_2.

For $n \geq 3$, let $s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$ and let T_n have sides s_n, s_n, s_{n-1}.

1. Prove that each triangle in the sequence has perimeter $a + b + c$.

2. Prove that for $n \geq 3$, $s_n - s_{n-1} = \dfrac{(-1)^{n-1}}{2^{n-1}}(s_1 - b)$.

3. What happens to the three sides of T_n as n increases without bound?

Solution 4

1. Let $P(n)$ be the proposition $2s_n + s_{n-1} = a + b + c$. We also define $s_0 = b$. Clearly $P(2)$ is true since $2s_2 + s_1 = a + b + c$. Let k be an integer for which $P(k)$ is true then

   $$2s_k + s_{k-1} = a + b + c.$$

   Now consider the proposition for $k + 1$,

   $$2s_{k+1} + s_k = (s_k + s_{k-1}) + s_k = 2s_k + s_{k-1} = a + b + c$$

   so $P(k + 1)$ and $P(k)$ are true $\forall k \geq 2$ by induction.

2.

$$\begin{aligned} s_n - s_{n-1} &= \frac{1}{2}(s_{n-1} + s_{n-2}) - s_{n-1} \\ &= (-\frac{1}{2})(s_{n-1} - s_{n-2}) \\ &= (-\frac{1}{2})^2(s_{n-2} - s_{n-3}) \\ &\vdots \\ &= (-\frac{1}{2})^{n-1}(s_1 - s_0) \\ &= (-\frac{1}{2})^{n-1}(s_1 - b) \end{aligned}$$

3. As $n \to \infty$, $s_n - s_{n-1} = 0$ and s_n is finite, x say, so that the sides s_n, s_n, s_{n-1} of the triangle $T(n)$ approach x, x, x, that is, the limiting triangle is equilateral.

Problem 5

For any positive integer n let $s(n)$ denote the digit sum and answer the following.

1. Show that for any positive integer k there is a positive integer m such that $m, 2m, \ldots, km$ all have digit sum divisible by 10.

2. Prove that for any positive integer m there is a positive integer k, for which $s(km)$ is not a multiple of 10.

Solution 5

1.

$$m = 10\cdots010\cdots01\cdots10\cdots01$$

with 10 ones and p zeros in each block of zeros between the ones where $k < 10^p$ is such a number.

2. Choose t such that $m < 10^t$. Then there exist multiples n_1 and n_2 of m such that

$$9 \times 10^{2t} \leq n_1 < 9 \times 10^{2t} + 10^t \quad \text{and} \quad 10^t \leq n_2 < 2 \times 10^t \ .$$

We can write decimal representations

$$n_1 = 9\overbrace{0\cdots0}^{t \text{ zeros}}a_1\cdots a_t\ , \quad n_2 = 1\,b_1\cdots b_t\ ,$$
$$n_1 + 10^t n_2 = 10\,b_1\cdots b_t\,a_1\cdots a_t\ ;$$

but then we have

$$\begin{aligned} s(n_1) &= 9 + a_1 + \cdots + a_t \\ s(n_2) &= 1 + b_1 + \cdots + b_t \\ s(n_1 + 10^t n_2) &= 1 + a_1 + \cdots + a_t + b_1 + \cdots + b_t \end{aligned}$$

so $s(n_1 + 10^t n_2) = s(n_1) + s(n_2) - 9$, and the three digit sums cannot all be multiples of 10.

Problem 6
Consider the list of fractions

$$\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots$$

that is,

$$\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \frac{1}{4 \times 5}, \ldots$$

with the list going on as far as we wish. Prove that if n is any integer greater than 1, then it is possible to select a finite sequence of consecutive terms from this list which add up to $1/n$.

Solution 6
Consider such a finite sequence

$$S = \sum_{j=n}^{n+m} \frac{1}{j(j+1)}$$

and note that

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}.$$

Then

$$\begin{aligned} S &= \sum_{j=n}^{n+m} \frac{1}{j} - \frac{1}{j+1} \\ &= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \ldots + \frac{1}{n+m-1} - \\ & \qquad \frac{1}{n+m} + \frac{1}{n+m} - \frac{1}{n+m+1} \\ &= \frac{1}{n} - \frac{1}{n+m+1} \\ &= \frac{m+1}{(n+m+1)n} \end{aligned}$$

Consider $m = n^2 - 1$ then

$$\begin{aligned} S &= \frac{n^2}{(n+n^2)n} \\ &= \frac{1}{1+n} \end{aligned}$$