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Solutions to Problems 1321-1330

Q1321 Find the sum of the coefficients of those terms in the expansion of

$$(x^{31} + x^5 - 1)^{2011}$$

which have an odd exponent in x.

ANS: Note that

$$(x^{31} + x^5 - 1)^{2011} = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $n = 31 \times 2011 = 62341$ is an odd number. Therefore, the sum of the coefficients of terms of odd exponent is

$$T = a_1 + a + 3 + \cdots + a_n.$$

Now let

$$P(x) = (x^{31} + x^5 - 1)^{2011}.$$

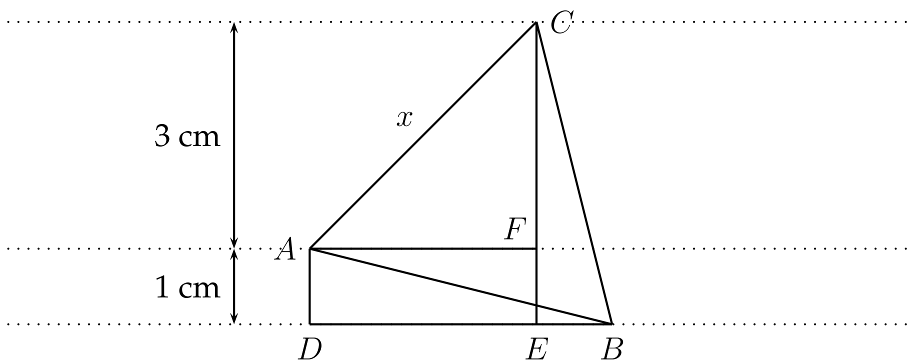
Then

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n = P(1) = 1$$
$$a_0 - a_1 + a_2 - a_3 + \cdots + a_{n-1} - a_n = P(-1) = (-3)^{2011} = -3^{2011}.$$

Hence

$$T = \frac{P(1) - P(-1)}{2} = \frac{1 + 3^{2011}}{2}.$$

Q1322 The vertices of an equilateral triangle lie on three parallel lines which are 3 cm and 1 cm apart, as shown in the figure. Find the side length of the triangle.



ANS: Let x be the side length. Then $BD^2 = x^2 - 1$ and $AF^2 = x^2 - 9$. Hence $BE = \sqrt{x^2-1} - \sqrt{x^2-9}$. Applying Pythagoras' Theorem to the triangle BCE gives

$$x^2 = 4^2 + (x^2-1) + (x^2-9) - 2\sqrt{(x^2-1)(x^2-9)},$$

or

$$2\sqrt{x^4 - 10x^2 + 9} = x^2 + 6.$$

By squaring both sides and simplifying we deduce

$$3x^4 - 52x^2 = 0.$$

Therefore, $x^2 = 52/3$ and thus $x = \sqrt{52/3}$.

Q1323 Let $f(x)$ be the sum of all terms in the expansion of $(x+a)^n$ which have an odd exponent in x, and let $g(x)$ be the sum of all terms which have an even exponent. Prove that

$$f^2(x) - g^2(x) = (x^2 - a^2)^n.$$

ANS: The question should be

$$f^2(x) - g^2(x) = -(a^2 - x^2)^n.$$

Note that

$$f(x) + g(x) = (a+x)^n, \quad f(-x) = -f(x) \quad \text{and} \quad g(-x) = g(x).$$

Hence

$$f^2(x) - g^2(x) = [f(x)+g(x)][f(x)-g(x)] = -[f(x)+g(x)][f(-x) \\ +g(-x)] = -(a+x)^n(a-x)^n = -(a^2-x^2)^n.$$

(Note: if n is odd then $f^2(x) - g^2(x) = (x^2-a^2)^n$. But if n is even then $f^2(x) - g^2(x) = -(x^2-a^2)^n$.)

Q1324 Jack wrote down all integers from 1 up to 999, in exactly that order. After writing down 540 digits, he stopped for a break. Which digit was the last one written?

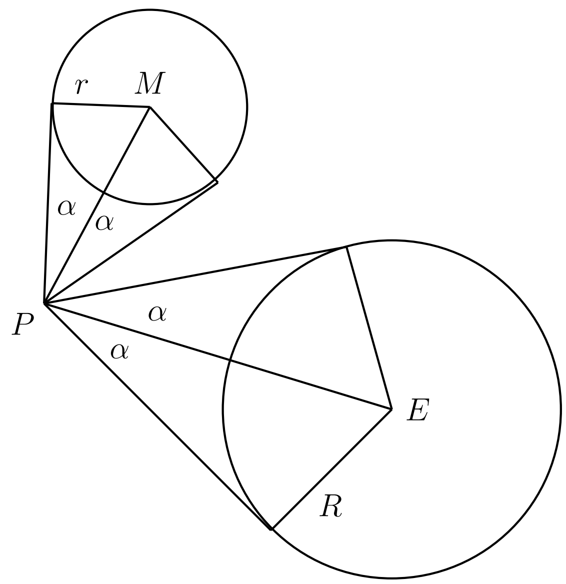
ANS:

$$\underbrace{1, 2, \ldots, 9}_{\text{9 digits}}, \underbrace{10, 11, \ldots, 99}_{90\times2=180\text{ digits}}, \underbrace{100, 101, \ldots, 199}_{100\times3=300\text{ digits}}, \underbrace{200, 201, \ldots, 215}_{16\times3=48\text{ digits}}.$$

After writing the integer 215, he has written 537 digits. The next number is 216, and the digit 6 is the 540^{th} digit.

Q1325 An astronaut is at a point P in space. At this point, the earth and the moon appear to be equally large to him. Let E and M be the centres, and R and r be the radii of the earth and the moon, respectively. Find PE/PM in terms of R and r.

ANS:



Since the earth and the moon appear equally large to the astronaut from P, they subtend equal angles at P. Therefore,

$$\sin\alpha = \frac{r}{PM} = \frac{R}{PE},$$

which implies

$$\frac{PE}{PM} = \frac{R}{r}.$$

Q1326 Find the value of

$$2010^2 - 2009^2 + 2008^2 - 2007^2 + \cdots + 4^2 - 3^2 + 2^2 - 1^2.$$

ANS: Let S be the sum. By grouping by pairs and factorising we obtain

$$\begin{aligned} S &= (2010 + 2009) + (2008 + 2007) + \cdots + (4 + 3) + (2 + 1) \\ &= 1 + 2 + \cdots + 2010 \\ &= \frac{2010}{2}(1 + 2010) = 2021055. \end{aligned}$$

Q1327 A quadrilateral $ABCD$ contains an inscribed circle C. Assume that the ratio of the area of Q to that of S is $4/3$. Find the ratio of the perimeter of Q to that of C.

ANS: Let $AB = a$, $BC = b$, $CD = c$, and $DA = d$ be the side lengths of Q, and let R be the radius of C whose centre is O. Then

$$\begin{aligned}\frac{\text{area}(ABCD)}{\text{area}(C)} &= \frac{\text{area}(\Delta(AOB)) + \text{area}(\Delta(BOC)) + \text{area}(\Delta(COD))}{\text{area}(C)} \\ &\qquad + \frac{\text{area}(\Delta(DOA))}{\text{area}(C)} \\ &= \frac{\frac{1}{2}(aR + bR + cR + dR)}{\pi R^2} \\ &= \frac{a+b+c+d}{2\pi R} \\ &= \frac{\text{perimeter}(ABCD)}{\text{perimeter}(C)}.\end{aligned}$$

Hence

$$\frac{\text{perimeter}(ABCD)}{\text{perimeter}(C)} = \frac{4}{3}.$$

Q1328 Find the maximum and minimum values of

$$A = \sin^3 x + \cos^3 x.$$

ANS: Since

$$A = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x) = (\sin x + \cos x)(1 - \sin x \cos x),$$

if we let $t = \sin x + \cos x$ then

$$A = \frac{1}{2}t(t^2 - 1).$$

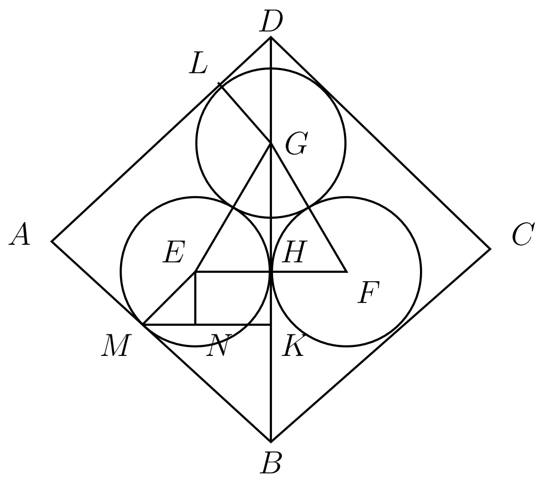
By using $ab + cd \leq \sqrt{a^2 + c^2}\sqrt{b^2 + d^2}$, we can see that $|t| \leq \sqrt{2}$, and therefore the problem reduces to finding the maximum and minimum values of

$$f(t) = \frac{1}{2}t(t^2 - 1) \quad \text{for } -\sqrt{2} \leq t \leq \sqrt{2}.$$

A study of the derivatives of f shows that

$$A_{\max} = f_{\max} = f(\sqrt{2}) = \frac{1}{\sqrt{2}} \quad \text{and} \quad A_{\min} = f_{\min} = f(\sqrt{-2}) = -\frac{1}{\sqrt{2}}.$$

Q1329 Three discs of radius 1 unit are contained in a square, as shown in the diagram. Find the side length of the square.



ANS: To find the side length AB, it suffices to find the length of the diagonal BD, which can be computed as

$$\begin{aligned}
BD &= DG + GH + HK + KB \\
&= DG + GH + EN + KM \\
&= DG + GH + EN + MN + NK \\
&= DG + GH + 2EN + EH \\
&= \frac{GL}{\sin 45^\circ} + \sqrt{EG^2 - EH^2} + 2EM\sqrt{\sin 45^\circ} + EH \\
&= \sqrt{2} + \sqrt{4-1} + \frac{2}{\sqrt{2}} + 1 \\
&= 2\sqrt{2} + \sqrt{3} + 1.
\end{aligned}$$

Hence

$$AB = BD \sin 45^\circ = 2 + \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}.$$

Q1330 For any natural numbers m and n, let

$$S_m(n) = 1^m + 2^m + \cdots + n^m.$$

Prove that

$$\binom{m+1}{0} S_0(n) + \binom{m+1}{1} S_1(n) + \cdots + \binom{m+1}{m} S_m(n) = (n+1)^{m+1} - 1,$$

where, for $0 \le k \le m+1$,

$$\binom{m+1}{k} = \frac{(m+1)!}{k!(m+1-k)!}.$$

ANS: The binomial formula gives, for all real values x and positive integers m,

$$(x+1)^{m+1} = \binom{m+1}{0} + \binom{m+1}{1}x + \binom{m+1}{2}x^2 + \cdots + \binom{m+1}{m+1}x^{m+1}.$$

Hence, with $x = 1, 2, \ldots, n$,

$$\begin{aligned}
2^{m+1} &= \binom{m+1}{0} + \binom{m+1}{1} + \binom{m+1}{2} + \cdots \\
&\qquad + \binom{m+1}{m+1} \\
3^{m+1} &= \binom{m+1}{0} + \binom{m+1}{1}2 + \binom{m+1}{2}2^2 + \cdots \\
&\qquad + \binom{m+1}{m+1}2^{m+1} \\
\vdots \quad & \quad \vdots \quad \vdots \\
(n+1)^{m+1} &= \binom{m+1}{0} + \binom{m+1}{1}n + \binom{m+1}{2}n^2 + \cdots \\
&\qquad + \binom{m+1}{m+1}n^{m+1}.
\end{aligned}$$

Adding columnwise gives

$$\begin{aligned}
S_{m+1}(n+1) - 1 &= \binom{m+1}{0}S_0(n) + \binom{m+1}{1}S_1(n) + \cdots \\
&\qquad + \binom{m+1}{m}S_m(n) + S_{m+1}(n),
\end{aligned}$$

or

$$\begin{aligned}
&\binom{m+1}{0}S_0(n) + \binom{m+1}{1}S_1(n) + \cdots + \binom{m+1}{m}S_m(n) \\
&= S_{m+1}(n+1) - S_{m+1}(n) - 1 \\
&= (n+1)^{m+1} - 1.
\end{aligned}$$