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The First Atomic Test

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Here I revisit a topic I have written on several times before, and which furthermore continues the theme of my last column. In the third issue of *Parabola Incorporating Function* in 2007, I discussed a technique called *dimensional analysis* and briefly touched on its application by G I Taylor to discover the energy released in the very first atomic blast (the so-called *Trinity Test*) in 1945: this happened at a time when that figure was still classified as top secret by the US military. I had earlier given a fuller account of the calculation involved in *Function* (February 1986, with a follow-up in February 1995). I also used this material in a set of course notes at Monash in 1996.

In all these, I was more concerned with the actual Mathematics rather than with the historical detail. Here by contrast, I want to examine that detail more carefully. The story is an intricate one and, as I unravel it, it will emerge that what I wrote earlier, and what is still popular belief, is not entirely accurate. Most of the material that is widely available gives a greatly simplified account of the full situation.

My own interest in the case began a long time before I wrote the articles for *Function*, and I'm no longer sure when it was that I first decided to look at the matter more closely. Over lunch back in the 1960s, a colleague told me that Taylor had made this calculation on the basis of a photograph published in *Life* magazine. He also asserted that the Soviet physicist Leonid Sedov had performed this same feat. I did look at the published accounts given by both Taylor and Sedov, and found that it was probably Sedov who first gave the simple version involving dimensional analysis, although he *did not* publish an estimate of the yield.

An almost identical simplification was also given in 1980 by George Bluman. Bluman is a co-author of an important text detailing the applications of Group Theory (a branch of abstract Algebra) to differential equations. Most scientific laws take the form of differential equations; they specify the *rates* at which various processes occur. Those rates are expressed as derivatives of functions we wish to discover. dimensional analysis is one aspect (an important aspect) of this use of Group Theory.

Furthermore, the Russian Grigory Barenblatt (who featured so prominently in my last column) also published an almost identical account in his book *Dimensional Analysis*.

In any case, the simplified account is the one you will find nowadays on numerous websites. To see it, Google **Dimensional Analysis Atom Bomb**. Most such versions credit the analysis to Taylor. However, as just indicated, this perpetuates a somewhat

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inaccurate account of the historical reality. My purpose here is to set the record straight, so here goes!

Let us start with Taylor. He himself wrote that in 1941 he was approached by the UK Ministry of Home Security, who had learned that "it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission". The ministry wanted to know what the effects of such a bomb would be.

G I (later Sir Geoffrey Ingram) Taylor (1886-1975) was one of the foremost English applied mathematicians of his day,[2] and it was natural that the Ministry should seek his advice. He went to work on the problem, and perhaps the most accessible account of the research it engendered is one by Barenblatt. In 1993, Barenblatt was appointed to a chair in Cambridge. It was one of those special chairs named in honour of some great figure, and in this instance that great figure was Taylor. According to custom, Barenblatt was required to deliver an "Inaugural Lecture" and the text of this was published by Cambridge University Press. In his lecture Barenblatt took some pains to relate the simplified account to what Taylor *actually did*. Here, however, I will rely more directly on Taylor's own account. It was produced, as I said, in 1941, that is to say, under wartime conditions. So it was not published at the time. In fact, he was not given clearance to publish it until 1949, when it formed the first of two papers both titled "The formation of a blast wave by a very intense explosion". They appeared in the *Proceedings of the Royal Society* in 1950 and this first one simply reprinted the original report to the ministry, apart from the provision of a brief introduction, a few other minor changes and the addition of some supplementary material at the end.

Taylor approximated the situation by supposing the explosion began from an initial volume so small that it could be regarded as a single point, and which gave rise to a spherical shock wave separating an inner fireball from the outside (normal) air. He then wrote down three partial differential equations to describe the situation. What goes on inside the fireball involves the pressure, the density and the radial velocity (i.e. the outward velocity field of the constituent particles). Each of these three quantities depends on the radial coordinate (that is to say, the distance from the initiating point inside the fireball) and also on the time since detonation; all three of the quantities of interest change with *both* of these independent variables. Because there are these *two* independent variables, with rates of change over both, the differential equations involved are what are called *partial* differential equations. These are much more complicated than the *ordinary* differential equations which involve but a single independent variable. The three equations he used were:

1. an equation of *motion* (essentially Newton's second law)

2. an equation of *continuity* (expressing the conservation of matter)

[2]One of my own teachers, the late Professor Sir Thomas Cherry, credited Taylor with the first significant validation of the Navier-Stokes equations, i.e. the equations governing the flow of viscous fluids. One of the million dollar prizes offered by the Clay Mathematics Institute is to be awarded for substantial progress toward the understanding of these equations. In *Function* (Feb 2004), I speculated that had the prize been on offer back in 1904, it might well have gone to Ludwig Prandtl for his development of Boundary Layer Theory; possibly Taylor's work could be put in a similar category!

3. an equation of *state* (relating density to pressure).

To produce a general solution of this set of equations is an almost impossible task, so what Taylor did was to look for a solution having a particular (relatively simple) form. Such solutions are termed *similarity solutions* and they do relate very closely to solutions produced by means of Group Theory and so to the combinations of variables derived from dimensional analysis. The simplified accounts take this as their starting point. Now if we apply dimensional analysis directly to the problem (without considering the partial differential equations at all!), we come up with a set of what are called *dimensionless quantities*: that is to say, quantities whose values are independent of the system of units we happen to use in our measurements. Different versions of the argument may have up to four of these, or else as few as two. However, the difference is immaterial as all but one can very plausibly be set equal to zero.

The remaining non-zero dimensionless quantity is $Et^2\rho^{-1}R^{-5}$, where:

E is the energy released by the explosion
t the time since detonation
ρ the density of the air outside the fireball
R the radius of the fireball.

A standard result of dimensional analysis (the Buckingham Pi Theorem, discussed in my earlier column) may now be applied to deduce a result that Taylor had earlier discovered, namely

$$\log R = \tfrac{2}{5}\log t + \tfrac{1}{5}\log\left(CE/\rho\right).$$

In this equation, the new symbol C stood for a simple numerical constant. Now Taylor certainly did produce this same equation, but there was one important difference. He had included in his analysis a further dimensionless quantity that the simplified version overlooks or simply neglects. This was a number usually represented by the symbol γ, derived from thermodynamics, relating to the third of his partial differential equations and connecting the pressure to the density of the hot gas inside the fireball. If we include this further quantity in the dimensional analysis, the constant C in the equation has to be replaced by a function of this further quantity γ. Taylor called this function *gotohere$S(\gamma)^5$. Thus we may recast the equation as

$$E = \tfrac{\rho R^5}{t^2 S(\gamma)^5} \qquad .$$

The two equations just given are really the same equation, but the second form makes it explicit that the value the constant assumes depends on the value assigned to γ. I will call this equation (in either of its forms) the *basic equation*. However we write it, it provides a connection between the physical quantities R, t and E (via ρ, whose value is assumed constant).

In Taylor's first paper, the basic equation (in a slightly amended version of the first form) is accorded considerable prominence, but the use to which it is put in that paper is not to afford a calculation of E but rather of R, which measures the extent of the devastation produced by the explosion. Taylor's rather surprising conclusion is that "an atomic bomb would be only half as efficient, as a blast-producer, as a high explosive releasing the same energy"[3]. (But, of course, high explosives *do not* release the same energy!)

Now Taylor's first paper discusses a lot more than this. It deduces the distribution of pressure, density and radial velocity within the fireball, but also goes on to look at temperature and energy, and pays particular attention to the edge of the fireball, that, say the shock-wave itself. The big uncertainty in his computations, which were carried out by means of laborious approximations, was the value to assign to γ. In everyday conditions, $\gamma = 1.4$, but it was by no means certain that this same value would prevail in the extreme conditions encountered inside the fireball. Nonetheless, he based most of his calculations on the value 1.4, although he also paid attention to another different possibility: $\gamma = 1.667$.[4] What Taylor *did not* pay much attention to in that first paper was the direct calculation of $S(\gamma)$. It is implicit in what he wrote that it *could* be calculated, but that calculation was not presented explicitly (although closely related calculations are). For the full elaboration of this aspect of the story, we have to wait for Part II of Taylor's paper on blast waves.

However, before we get to that, let us look at some more of that early history. As we have seen, Part I of Taylor's paper was prepared in 1941. When it was published (in 1950), it carried the sub title *Theoretical Discussion*. But Taylor was not alone in his interest in the problem. What was going on at much the same time is well summarized by George Batchelor [5] in his book *The Life and Legacy of G. I. Taylor*. There he tells of parallel "theoretical discussions" by John von Neumann[6] in the USA and Leonid Sedov in the Soviet Union. von Neumann, working directly for the US military, produced a report only a very short while after Taylor completed his. In fact, Taylor's analysis was submitted on 27 June 1941 quite coincidentally only three days ahead of von Neumann's! [The delay was caused by von Neumann's taking a weekend to recheck his calculations.] Like Taylor's, however, his analysis was not made public until much later. This was also very possibly true of Sedov's work, which was first published in 1946. (So it was in fact Sedov whose account first appeared, albeit in Russian, in the open literature. This claim is actually implicit in a footnote in Sedov's subsequent book.) All these authors used the same approximation, replacing the focus of the explosion by a mathematical point. The resulting solution is sometimes given the names of all three of them, but is also known as the *point source solution*.

Von Neumann, unlike Taylor, replaced the set of three partial differential equations at the outset by their simplified form as ordinary differential equations arising from a similarity assumption, and furthermore was able to simplify one, replacing the equation of motion by an energy integral. He also motivated the quest by an explicit search

[3] This is because much more of the energy is dissipated as heat.

[4] Air is almost entirely made up of oxygen (O_2) and nitrogen (N_2), both *diatomic gases* (i.e. each molecule contains two atoms). The theoretical value of γ for a diatomic gas is $7/5$ ($= 1.4$). For a monatomic gas, the theoretical value is $5/3$ ($= 1.666...$). Taylor was entertaining the possibility that with the release of atomic energy, the molecules became disassociated, and so the gas became monatomic.

[5] George Batchelor (1920-2000) was an expatriate Australian, who studied under Taylor, and subsequently became head of Applied Mathematics at Cambridge.

[6] This is the same John Von Neumann mentioned briefly in my column on Babbage and the computer.

for dimensionless quantities, although he did not explicitly produce the "simplified version". Because of his simpler enunciation of the problem, he was able to provide exact solutions in places where Taylor had had to resort to approximate methods.

Although Sedov first published his analysis in 1946, it is reasonable to assume that in this case also the work was done rather earlier but likewise delayed by wartime protocols. We don't know this for certain because Sedov makes no mention of military applications of his work. The most accessible account of that work is the one given in his book, titled in a 1959 English translation, *Similarity and Dimensional Methods in Mechanics*.

Taylor, von Neumann and Sedov were all well aware that their analysis was an approximation applying to a certain restricted time frame. For *very short* times, it is invalid to treat the source of the blast as a simple point; for somewhat longer times, other processes become involved and again the analysis breaks down as these other physical processes become important, producing the well-known "mushroom cloud" – in particular, the basic equation ceases to be valid. This is exactly where Barenblatt's "intermediate asymptotics" enter the picture. The basic equation applies for a (really very short) range of intermediate times. In his book on this subject, he goes into some detail on the range over which these apply in this very case.

But let us now turn to G I Taylor's second paper. This was explicitly concerned with the first test of an atom bomb in New Mexico on 16 July, 1945. As part of the testing process, a film was taken, whose successive frames displayed the fireball as it grew in time, and various other pictures were also taken at precisely known times. This material was declassified in 1947, and still photographs from it were indeed widely published, in *Life* magazine and elsewhere. Taylor examined the full sequence of pictures (25 in all), each of which came supplied with a precise time and also an accurate length scale.

If we go back to the first form of the basic equation, we see that a plot of $\frac{5}{2}\log R$ against $\log t$ should give a straight line, with slope 1. The different frames provided 25 pairs of values of R and t. They cover the times between $t = 0.1$ millisecond to $t = 62$ milliseconds. The 25 points lie, all but one, on an almost perfect straight line with exactly the right slope. The one exception is the very first, when t was too small for the point source approximation to be valid. The graph Taylor drew thus provided a triumphant confirmation of the underlying theory. Such experimental validation was needed because it was by no means clear that γ might not have varied with time. Had it done so, then the basic equation might not have applied, and the theoretical discussion could well have proved inadequate.

But once the underlying model was vindicated, the way was open for Taylor to calculate the value of E. He was still unsure what value to give to γ, so he made four different calculations for the values 1.200, 1.300, 1.400 and 1.667. He calculated a quantity $K(\gamma)$ for all four values of γ, and gave a third form of the basic equation

$$E = K(\gamma)\rho R^5 t^{-2}.$$

(This is simply to call the function *gotohere $S(\gamma)^{-5}$ by a new, more convenient name; it is also the reciprocal of C of the first form, and again making explicit the dependence on γ.) He made detailed calculations for each of his values, and converted each into an estimate of E in a value expressed in kilotons of TNT. He thus had four estimates, which were respectively: 34.0, 22.9, 16.8 and 9.5 kilotons. He paid particular attention to the cases $\gamma = 1.4$ and $\gamma = 1.3$, and ended up preferring the value $\gamma = 1.4$, and so estimating E as 16.8 kilotons.[7] This corresponds to the value $K(1.4) = 0.856$. Von Neumann calculated the same value and put it at 0.851, which is marginally more accurate. Sedov proceeded along slightly different lines and calculated the reciprocal of $K(\gamma)$ (i.e. Taylor's $S(\gamma)^5$) which he called $\alpha(\gamma)$. He found $\alpha(1.4) = 1.175$, which corresponds exactly to Von Neumann's figure of $K(1.4) = 0.851$, although in his book he went on to quote Taylor's estimate of 16.8 kilotons.

There seems, however, to be an unspoken consensus that these values are too low. Barenblatt, in his inaugural lecture, states that Taylor calculated $K(1.4) = 1.033$. He took this figure from a textbook by Zel'dovich and Raizer, and speculates that they produced it by back-calculation from what they thought as the "right answer". However, on Taylor's figures, this would correspond to a value $\gamma \approx 1.34$, and on Von Neumann's, very accurately, to $\gamma = 1.3342$. Today, many websites either explicitly or implicitly take the value to be approximately 1. However, if we use the 1.033 figure for $K(1.4)$ instead of Taylor's, then the estimate of the yield is altered to 20.3 kilotons, but the argument has become circular!

We should also take note of an observation by Taylor, Batchelor and Sedov to the effect that the energy calculated is "more precisely, the part of the energy not radiated outside the wave front". In a footnote, Sedov notes that "it should be kept in mind that a considerable part of the energy is expended in radiation". Taylor states that he is calculating "that part of the energy which was not radiated outside the ball of fire"; the part that *was* so radiated he says "cannot be estimated".

So, how good is this estimate? To answer this question, we need to know what the yield *actually was* for the Trinity Test. This figure was still classified when Taylor wrote, and it may well be that it came as a surprise to the US military that the secret figure was calculable from the material they had already released. (However, had they bothered to ask von Neumann, they would have discovered that they had no cause for surprise! von Neumann had also deduced a form of the basic equation, and if we apply this to his value of $K(1.4)$, we obtain an estimate of 17.4 kilotons for the yield.)

I have tried to find what the actual value was, and that number is not easy to determine exactly. The Wikipedia article **Nuclear weapon yield** [accessed 16 October 2010] contains a table that gives the value 19 kilotons for this bomb, but later, in an account of Taylor's calculation, that very same article says he estimated 22 kilotons, when the exact value was 20. It also repeats the story about the *Life* magazine picture. However, this same paragraph gives erroneous values of ρ and also of the fireball's radius at time 25 milliseconds, although it does point to the difficulty of measuring yields very accurately.

Batchelor tells us that President Truman announced a figure of 20 kilotons (presumably after the yield was declassified). He also informs us that Taylor "was mildly

[7] There is actually a minor error in this calculation. More correct is 17.5 kilotons.

admonished by the US army for publishing his deductions from their (unclassified) photographs". The military commander of the project, General Leslie Groves, reported that the yield was "the equivalent of $15,000$ to $20,000$ tons of TNT".

But now there is another detail to be cleared up. Many accounts of the simplified (dimensional analysis) calculation tell us that the correct value of $K(\gamma)$ was determined by experiments with conventional explosives. Bluman claims this in his article, as did I in my articles in *Function* and *Parabola*. However, the account I have just given makes it clear that Taylor *did not* take this route. In his first paper, he included a discussion on "actual high explosives", and this matter was taken further in his supplementary addendum, added when more powerful such explosives had become available. The problem with this route is that to produce a powerful explosion with conventional explosives, we need to use quite a lot of them, and this calls into serious question the validity of the point-source approximation. Even in the range where the comparison might be applicable, there were various serious discrepancies between the theoretical and the observed values of the pressure.[8]

So Taylor most certainly did not use such an argument. Both Bluman and I erred when we said that he did. It is an easy trap to fall into. dimensional analysis can reduce the problem to an application of the basic equation, and all we need to know for this purpose is the value of a single numerical constant, $K(1.4)$. In many other applications of Dimensional Analysis, such a value could, and indeed would, be determined by a single experiment. However, this was not such a case here. Taylor *calculated* the value he needed.

Finally, let us ask why Taylor has his name especially associated with this work while von Neumann and Sedov are less celebrated. Batchelor has surely given the definitive answer to this question. While all three developed the point source solution, it was only Taylor who thought to submit it to experimental verification and succeeded so triumphantly!

References and Further Reading

There is a lot written on this question, and I hope I have made it clear that it is not all historically accurate. Even when we have good reason to trust a source, it pays to be wary. For example, one of Taylor's versions of the basic equation is marred by a minor misprint, and I have mentioned also a small arithmetic error in one of his calculations; there are a couple of slight inaccuracies in Batchelor's account, and I have, in the body of the article, given details of more serious errors in other works. These cautions should be borne in mind in reading the source material. I summarize it here, but with the caution that this is a *select* bibliography; there is a lot more relevant material available than I list here.

Taylor's two papers are:

[8] Sadly, we see confirmation of this on our television sets as the news all too frequently carries graphic portrayals of explosions. The explosives that cause them are often placed in cars and trucks, which are not point sources. The resulting fireballs are not spherical.

- "The Formation of a Blast Wave by a very Intense Explosion: I. Theoretical Discussion"
- "The Formation of a Blast Wave by a very Intense Explosion: II The Atomic Explosion of 1945",

These were published in the *Proceedings of the Royal Society* (of London), *series* **A,** in Volume **201** (1950), pp. 159-174, 175-186, but are now more readily accessed in *The scientific papers of Sir Geoffrey Ingram Taylor, Volume* **3** (edited by G. K. Batchelor), Cambridge University Press, pp. 493-509, 510-521. As noted in the text, the first of these was essentially a report to the British Government, submitted on Friday 27/6/1941.

Von Neumann's account is now available in his book:

- *Collected Works*, *Volume* **6** (edited by A. H. Taub) (New York, Pergamon Press,1963), pp. 219-237.

This is obviously a revised version of the original report, submitted to the US authorities on Monday 30/6/1941, but subsequently amended to include references to Taylor and to others who had worked on the problem.

Sedov's work first appeared in a Russian journal, whose title translates as *Applied Mathematics and Mechanics* (Volume **10** (1946), pp. 241-250). However, this material is more readily available in his book:

- *Similarity and Dimensional Methods in Mechanics* (London: Infosearch, 1959). See especially pp. 210-215.

This is an English translation of the fourth Russian edition, so the first edition would have been published considerably earlier. However, it may have been the third edition that first carried his account of Taylor and the bomb.

Batchelor's biography of G I Taylor is:

- *The Life and Legacy of G. I. Taylor* (Cambridge University Press, 1996); see especially pp. 202-207.

It is this source that attributes the 20 kiloton figure for the yield to an announcement by President Truman. There is also a biography of Taylor posted on the St Andrews site. Go to www-groups.dcs.st-and.ac.uk/~history/Biogindex and follow the prompts.

Barenblatt's accounts are to be found in:

- *'Scaling Phenomena in Fluid Mechanics'* (Cambridge University Press, 1994)
- *Dimensional Analysis* (New York: Gordon and Breach, 1987). See especially pp. 41-44.

The account in this latter book duplicates another version, which however goes into greater detail on the role played by "intermediate asymptotics" in the validity of the basic equation. See Barenblatt's book:

- *Scaling, Self-Similarity and Intermediate Asymptotics* (Cambridge University Press, 1996), p. 90.

Bluman's telling of the story forms part of his paper:

- "Dimensional Analysis, Modelling and Symmetry"

This was published in the *International Journal of Mathematical Education in Science and Technology, Volume* **14** (1983), pp. 259-272, but see especially pp. 265-270. It was the published text of a talk delivered on 12/12/1980.

A biography of Batchelor is available on the web at the St Andrews site detailed above.

Zel'dovich & Raizer's text is:

- *Physics of Shock Waves and High-Temperature Phenomena, Volume* **1** (New York and London: Academic Press, 1966).

My own previous accounts of this material are:

- "The Mathematics of Measurement" *Function* **10(1)** (1986), pp. 14-22.
- "The First Atom Bomb" *Function* **19(1)** (1995), pp. 20-22.
- "Physical Mathematics" *Parabola* **43(3)** (2007), pp. 2-11.

General Groves' report is available on the web at

- http://www.atomicarchive.com/Docs/Trinity/Groves.shtml