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The Body at the Bottom of the Cliff

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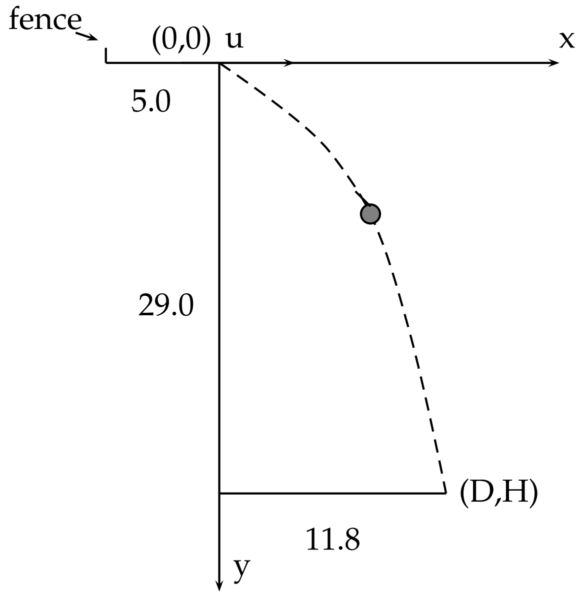
In the latter part of year 2009, I attended a scientific talk at Sydney University about the path of a body exiting a cliff. The position at which the body lands from the base of the cliff depends on exit velocity. We show here that for small exit velocities and small cliff heights the landing position is not impacted significantly by the angle of inclination and air resistance.



Letting the origin $(0,0)$ be the point where the body exited the cliff, the equations of motion

$$(\ddot{x}\ \ddot{y}) = (0,\ g)$$

where g is the acceleration due to gravity and double dots denote double differentiation with respect to time t, lead to

$$(D,H) = \left(Tu, \frac{1}{2}gT^2\right) \tag{0.1}$$

where $T(s)$ is the flight time from $(0,0)$ to (D,H) and $u(ms^{-1})$ is the body's exit velocity, so that

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$$(T, u) = \left(\sqrt{\frac{2H}{g}}, D\sqrt{\frac{g}{2H}} \right). \tag{0.2}$$

Here H is the height of the cliff and D is the horizontal distance from the base of the cliff. The case under consideration, namely $(D, H) = (11.8, 29.0)$, yields $(T, u) = (2.43, 4.85)$.

a) The effect of take-off inclination

If θ is the take-off angle above the horizontal then

$$(\dot{x}, \dot{y}) = (u\cos\theta, gt - u\sin\theta)$$

and equations (1),(2) are extended to

$$(D, H) = (Tu\cos\theta, \frac{1}{2}gT^2 - Tu\sin\theta)$$

$$(T, u) = \left(\sqrt{\frac{2(H + D\tan\theta)}{g}}, \frac{D}{\cos\theta}\sqrt{\frac{g}{2(H + D\tan\theta)}} \right).$$

With the use of differential calculus, it can be shown that the minimum required value of u for the body to reach (D, H), say $\tilde{u}$, corresponds to the angle

$$\tilde{\theta} = \frac{1}{2}\tan^{-1}(\frac{D}{H})$$

depicted below with the use of a ruler and set square, yielding

$$(\tilde{T}, \tilde{u}) = \left(\sqrt{\frac{2\sqrt{H^2 + D^2}}{g}}, \sqrt{g(\sqrt{H^2 + D^2} - H)} \right).$$

For the case under consideration

$$\tilde{\theta} = 11^\circ, (\tilde{T}, \tilde{u}) = (2.53, 4.76)$$

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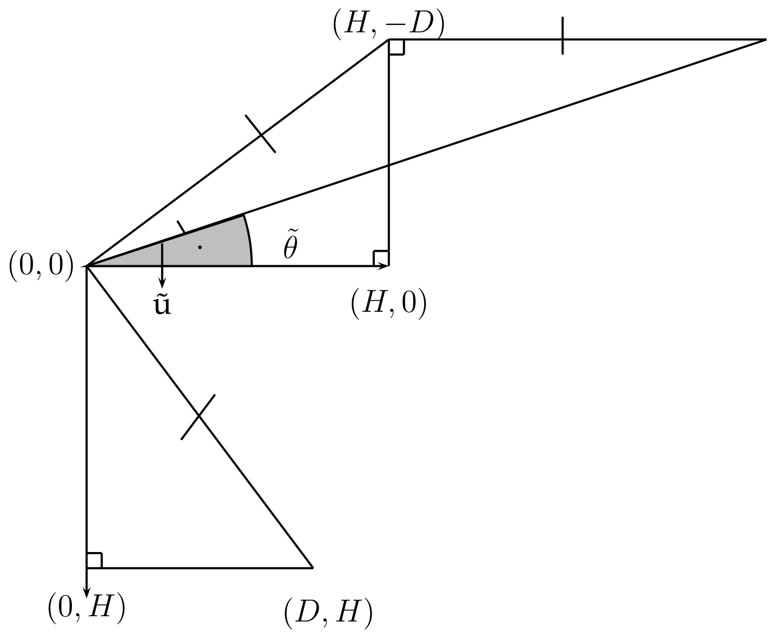

b) The effect of air friction

Assuming the coefficient of friction ε to be small, the equations of motion are extended to

$$(\ddot{x},\ddot{y}) = \left(-\varepsilon\dot{x}^{1+\delta}, g-\varepsilon\dot{y}^{1+\delta}\right)$$

where δ is a given constant, usually 0 or 1, so that (1),(2) are extended to

$$(D,H) = \left(Tu(1-\frac{\varepsilon T u^{\delta}}{2}), \frac{1}{2}gT^2(1-\frac{2\varepsilon}{(2+\delta)(3+\delta)}g^{\delta}T^{1+\delta})\right)$$

$$(T,u) = \left(\sqrt{\frac{2H}{g}}(1+\varepsilon r), D\sqrt{\frac{g}{2H}}(1+\varepsilon s)\right)$$

$$r: \ = \frac{(2H)^{\delta}}{(2+\delta)(3+\delta)}\left(\sqrt{\frac{2H}{g}}\right)^{1-\delta}$$

$$s: \ = \frac{1}{2}\left(\sqrt{\frac{2H}{g}}\right)^{1-\delta}\left(D^{\delta}-\frac{2}{(2+\delta)(3+\delta)}(2H)^{\delta}\right).$$

An estimate for ε can be derived in terms of T_o, the time taken by a similar body for a vertical fall from $(0,0)$ to $(0,H)$, as

$$\varepsilon = \frac{(2+\delta)(3+\delta)}{2g^{\delta}T_o^{1+\delta}}\left(1 - \frac{2H}{gT_o^2}\right)$$

while an estimate for δ is given by

$$\left(\hat{T}_o \,/\, T_o\right)^{1+\delta} = \frac{1 - 2\hat{H} \,/\, g\hat{T}_o^2}{1 - 2H \,/\, gT_o^2}.$$

where $\hat{T}_o$ is the time taken for an alternate vertical fall to $(0, \hat{H})$. For the case under consideration, for a landing at $D = 11.8$ metres from the base of the cliff and a fall of $H = 29.0$ metres, the table below shows how the time of flight T and the required initial velocity u depend on the air resistance parameterized by ϵ and δ.

δ	ϵ	(T, u)
0.0	.02	(2.45,4.93)
0.5	0.015	(2.48,4.95)
1.0	0.01	(2.55,4.90)

These results show that take-off inclination and air resistance may be overlooked if ϵ is small. This is to be expected in the case of a body falling through heights considered here. However in the case of a feather ϵ would be much larger than the values shown here and a quadratic approximation (including a term ϵ^2) for (D, T) would provide more accuracy in this case.