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Solutions 1351–1360

Q1351 A city consists of a rectangular grid of roads, with m roads running east–west and n running north–south. Every east–west road intersects every north–south road. A construction vehicle travels around the city, visiting each intersection once (and only once) and finally returning to its starting point. As it travels it builds a fence down the middle of each road it uses: thus it constructs, in effect, a single long fence which eventually loops back on itself. How many city blocks are now inside the fence?

SOLUTION If we assume that each block is a square with side length 1 unit, then the number of blocks inside the fence is equal to the area enclosed by the path. Since the path is a polygon with lattice points for its vertices we can use Pick's Theorem:

$$A = I + \tfrac{1}{2}P - 1 \ ,$$

where A is the area enclosed, I is the number of points inside the path and P is the number of points on the path. In this case there are mn points altogether, and they are all on the path (because we are told that every intersection was visited). Therefore $P = mn$ and $I = 0$, and the number of blocks inside the path is $\frac{1}{2}mn - 1$.

Q1352 Find infinitely many triangles with integer side lengths which contain an angle of 120°.

SOLUTION If the side lengths are x, y, z with the 120° angle opposite side z, the cosine rule gives

$$z^2 = x^2 + y^2 + xy \ .$$

The best way to deal with this is to write it as

$$(2z)^2 - (2y + x)^2 = 3x^2 \ ;$$

we can find infinitely many solutions by letting x be odd and noting that an odd number is always the difference of two squares,

$$2m + 1 = (m + 1)^2 - m^2 \ .$$

If $x = 2m + 1$ we have $3x^2 = 2(6m^2 + 6m + 1) + 1$ and so we want

$$2z = 6m^2 + 6m + 2 \ , \quad 2y + x = 6m^2 + 6m + 1 \ ;$$

solving gives

$$x = 2m + 1 \ , \quad y = 3m^2 + 2m \ , \quad z = 3m^2 + 3m + 1$$

for $m = 1, 2, 3, \ldots$.

Comments. Taking $m = 1$ we find that a triangle with sides $3, 5, 7$ includes a 120° angle

– not a result which everyone knows! Note that we have found infinitely many possibilities, as the question asked, but we have not found *all* possibilities. For example you may check that the triangle with sides $7, 8, 13$ includes a $120°$ angle but does not fit into the above pattern.

Correct solution received from John Barton, Victoria.

Q1353

(a) On a $4 \times n$ chessboard we wish to place $2n$ knights in such a way that none attacks any other. Give three possible ways of doing this.

(b) Prove that there is no closed knight's tour on the $4 \times n$ board.

SOLUTION

(a) A knight on a chessboard always moves to a square of a different colour. So if we put knights on the $2n$ white squares of a $4 \times n$ board, none will attack another. A second solution, of course, is to put the knights on the black squares. A third solution is to place them on the two sides of length n: since there are two empty rows between them, the knights on one side cannot "reach" those on the opposite side.

(b) Now suppose that there is a closed knight's tour on the $4 \times n$ chessboard, and consider how we can use this tour to locate $2n$ non–attacking knights. Since no two consecutive squares on the tour can be occupied by non–attacking knights, there are only two possible placements for the $2n$ knights: on every second square of the tour, or on every *other* second square of the tour. But we know that there are in fact three ways to place the knights (perhaps more): the only possible conclusion is that the closed knight's tour on the $4 \times n$ board cannot exist.

Q1354 A sequence $a_1, a_2, a_3, \ldots$ of positive integers has the properties

$$a_n^2 - a_{n-1}a_{n+1} = 1$$

for all $n \geq 2$, and $a_1 = 1$.

(a) Prove that a_2 cannot equal 1.

(b) Prove that if $a_2 = 2$ then $a_n = n$ for all n.

(c) Prove that if $a_2 \geq 3$ then $a_n > a_{n-1} + 1$ for all $n \geq 2$.

(d) Find all values of a_2 and all values of n such that $a_n = 2011$.

SOLUTION

(a) If $a_2 = 1$ we immediately get $a_3 = 0$ which is not a positive integer.

(b) The proof is by mathematical induction. If $a_2 = 2$ then $a_n = n$ for $n = 1, 2$; if $n \geq 3$ and the result is true for a_{n-1} and for a_n then

$$a_{n+1} = \frac{a_n^2 - 1}{a_{n-1}} = \frac{n^2 - 1}{n - 1} = n + 1 \, .$$

(c) Proof by induction. With $a_2 \geq 3$ it is clearly true that $a_n > a_{n-1} + 1$ for $n = 2$. If the inequality is true for some specific $n \geq 2$ then

$$a_{n+1} = \frac{a_n^2 - 1}{a_{n-1}} = (a_n + 1)\frac{a_n - 1}{a_{n-1}} > a_n + 1 \ .$$

(d) From (b) it is clear that one solution is $a_2 = 2$, $a_{2011} = 2011$; since a_2 can be any positive integer except 1, another solution is $a_2 = 2011$. We shall prove that there are no further solutions. So, suppose that $a_2 \geq 3$ and $n \geq 3$. Then we have

$$a_n a_{n-2} = a_{n-1}^2 - 1 = (a_{n-1} + 1)(a_{n-1} - 1)$$

and so a_n is a factor of $(a_{n-1} + 1)(a_{n-1} - 1)$. But if $a_n = 2011$, which is a prime number, this means that either

$$a_n \mid a_{n-1} + 1 \quad \text{or} \quad a_n \mid a_{n-1} - 1 \ ;$$

since $a_n > a_{n-1} + 1$ this is impossible. (The notation $x \mid y$ means that x is a factor of y.)

Q1355 Let n be a positive integer and suppose that 2^n and 5^n begin with the same digit. Prove that there is only one possibility for this digit.

SOLUTION Suppose that the number 2^n consists of a digit a followed by s further digits and 5^n consists of a followed by t further digits. Then we have

$$2^n = (a + x)10^s \quad \text{and} \quad 5^n = (a + y)10^t \ ,$$

where $0 \leq x < 1$ and $0 \leq y < 1$. Multiplying these equations,

$$10^n = (a + x)(a + y)10^{s+t}$$

and so

$$(a + x)(a + y) = 10^{n-s-t} \ .$$

However $n - s - t$ is an integer, and $1 \leq (a + x)(a + y) < 100$, so either

$$(a + x)(a + y) = 1 \quad \text{or} \quad (a + x)(a + y) = 10 \ .$$

In the first case we have $a = 1$, $x = 0$, $y = 0$, so $2^n = 10^s$; since n is a positive integer, this is impossible. In the second case we have

$$a^2 \leq (a + x)(a + y) < (a + 1)^2 \ ,$$

that is,

$$a^2 \leq 10 < (a + 1)^2 \ ,$$

and so $a = 3$ is the only possible digit. And indeed, if $n = 5$ then $2^n = 32$ and $5^n = 3125$, both of which start with a 3.

Comment. In fact, it is possible to show that there are infinitely many such n: the numbers 2^n and 5^n both begin with 3 for $n = 5, 15, 78, 88, 98, 108, 118, \ldots$ (however the obvious pattern does not continue).

Partial solution received from John Barton, Victoria.

Q1356 Let $N = 11!$ How many positive cubes divide N?

SOLUTION We have

$$\begin{aligned} 11! &= 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 11 \times 7 \times 5^2 \times 3^4 \times 2^8 \\ &= 11 \times 7 \times 5^2 \times 3 \times 2^2 \times 12^3 \ . \end{aligned}$$

Therefore n^3 divides $11!$ if and only if n divides 12: there are six possibilities, namely, 12^3, 6^3, 4^3, 3^3, 2^3 and 1^3. **Now try** problem 1368 in this issue.

Q1357 Find the sum of the digits of the number

$$N = (2011^2)(625^{2011})(4^{4022}) \ .$$

SOLUTION We have

$$\begin{aligned} N &= (2011^2)(625^{2011})(4^{4022}) \\ &= (2011)^2 \times (10000)^{2011} \\ &= 40441210000 \cdots 0000 \end{aligned}$$

and the sum of the digits is 16.

Correct solution received from John Barton, Victoria.

Q1358 Suppose that $\log(x^2y^{2n}) = 1$ and $\log(x^{2n}y^2) = 1$ where x and y are positive numbers and n is a positive integer. Show that

$$\log(x^ny^n) < 1 \ .$$

SOLUTION Adding the given equations and remembering the identity $\log X + \log Y = \log(XY)$ gives

$$\log(x^{2n+2}y^{2n+2}) = 2 \ .$$

Using $\log(X^m) = m \log X$, we have

$$(2n+2)\log(xy) = 2$$

and therefore

$$\log(x^ny^n) = n\log(xy) = \frac{2n}{2n+2} < 1 \ .$$

Correct solution received from John Barton, Victoria, who also proved from the given equations that x and y must be equal.

Q1359 Simplify

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{2011}+\sqrt{2010}} \ .$$

SOLUTION For any positive real number x we can rationalise the denominator to get

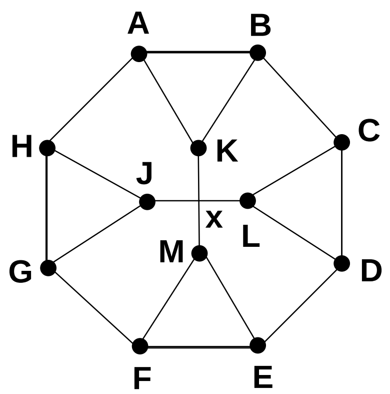
$$\frac{1}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{x+1}+\sqrt{x}}\,\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} = \sqrt{x+1}-\sqrt{x}\ .$$

Therefore

$$\begin{aligned}&\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2011}+\sqrt{2010}}\\ &\quad=\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\cdots+\sqrt{2011}-\sqrt{2010}\\ &\quad=\sqrt{2011}-1\ .\end{aligned}$$

Correct solution received from John Barton, Victoria.

Q1360 Shown below is a map of walking paths around a garden. Paths meet at the points labelled A to M. Note that x is not a path intersection but a place where one path runs along a bridge over another. Bridget is standing on top of the bridge at x and wishes to take a walk around the garden, visiting every path intersection exactly once and returning to x. In how many ways can she do this?



SOLUTION Clearly Bridget's walk must include the path JL along the bridge. Continuing from L she would have to take either LC or LD; for a start let's assume that she takes LC. Since she visits each intersection once only she can never use the path LD, as this would entail a repeat visit to L; since she must visit D, her path must include both CD and DE. Now she cannot take BC and hence must use both of AB and BK; it follows that she must at some stage walk along the path KM below the bridge, and along AH.

Now Bridget cannot take path HJ: if she did, GH and GJ would both be ruled out and she could never visit G (or if she did, she could never get back to the start). So she must take paths GH and GJ; for similar reasons she must take EF and FM, and this completes her tour of the garden. We have found only one possible route; but if we had assumed initially that she took LD instead of LC we would have found another. Also, each of these circuits could have been walked in either direction. So there are four possibilities altogether.