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Integrating expressions containing inverse functions

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The problem of integrating simple expressions containing inverse functions relies on the well-known method of integration by parts. For example, consider the integral

$$\int \ln x \, dx.$$

Since the natural logarithmic function is the inverse of the exponential function, the standard approach proceeds by recognising that the unit function $f(x) = 1$ can always be written as a product with the inverse function before the method of integration by parts is employed. In the case of the example given above, integration by parts leads to the well-known result of

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - x + C.$$

For convenience, in the remainder of the paper we will drop the arbitrary constant of integration appearing in all indefinite integrals.

As a second example, consider the integral

$$\int \sin^{-1} x \, dx.$$

Here the integrand of the integral consists of the inverse sine function and is the inverse of the well-known sine function. Again, using integration by parts for its evaluation we have

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} dx.$$

The integral appearing to the far right can be evaluated using the substitution $u = 1 - x^2$. Here $xdx = -du/2$, so that

$$\begin{aligned} \int \sin^{-1} x \, dx &= x \sin^{-1} x + \frac{1}{2} \int \frac{du}{\sqrt{u}} \\ &= x \sin^{-1} x + \sqrt{u} \\ &= x \sin^{-1} x + \sqrt{1 - x^2}. \end{aligned}$$

Evaluation of the integrals given in the above two examples using integration by parts is widely known and is nothing new in itself. In fact, these examples are typical

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of the type a pupil studying em Mathematics Extension 2 in NSW would be expected to know how to do and it is the above method they would be expected to use. As a technique for integrating relatively simple expressions containing inverse functions this "standard" method is more than adequate. However, for the integration of more complicated expressions containing inverse functions use of the standard method proves difficult. As an example, consider the evaluation of $\int (\sin^{-1} x)^3 dx$. Such integrals can however be evaluated in a relatively straightforward manner using a little known result I am going to call *Parker's method*.

Let us begin by considering, in simplest form, the evaluation of integrals of the form $\int f^{-1}(x)\, dx$. If we use the substitution $y = f^{-1}(x)$, then $x = f(y)$ so that $dx = f'(y)dy$. Thus

$$\int f^{-1}(x)\, dx = \int y f'(y) dy = y f(y) - \int f(y)\, dy,$$

upon integrating by parts.

The above technique allows for a simpler evaluation of integrals containing a single inverse function compared to the standard method. As an example, let us consider the evaluation of $\int \sin^{-1}(x)\, dx$ using this alternative technique. Firstly, if we let

$$y = f^{-1}(x) = \sin^{-1}(x),$$

so that

$$x = f(y) = \sin(y),$$

on using

$$\int f^{-1}(x)\, dx = y f(y) - \int f(y)\, dy,$$

substituting gives

$$\begin{aligned} \int \sin^{-1}(x)\, dx &= y \sin(y) - \int \sin(y)\, dy \\ &= y \sin(y) + \cos(y) \\ &= x \sin^{-1}(x) + \sqrt{1 - x^2}, \end{aligned}$$

as expected. Here use of a right-angled triangle which has y as one of its acute angles, so that $\sin(y) = x$, has been used in order to write $\cos(y)$ in terms of x. Whilst this alternative method may not seem that much simpler compared to the standard method it is now only a simple step to generalize the above technique to more complicated expressions containing inverse functions. As we shall see, this is where the real usefulness of the technique lay over that of the standard approach.

In generalizing the procedure, let us consider the evaluation of integrals of the form $\int G(x, f^{-1}(x))\, dx$. If we again use the substitution of $y = f^{-1}(x)$ so that $x = f(y)$, then $dx = f'(y)dy$. Thus

$$\begin{aligned} \int G(x, f^{-1}(x))\, dx &= \int G(f(y), y) f'(y)\, dy \\ &= G(f(y), y) f(y) - \int f(y) \frac{d}{dy} G(f(y), y)\, dy \qquad (0.1) \end{aligned}$$

upon integrating by parts.

Initially the integral appearing to the right of equation (0.1) may not seem any simpler than the original integral. However, as we shall shortly see, in many instances equation (0.1) does indeed represent a significant simplification in the evaluation of integrals containing more complicated inverse function expressions. The technique is not new. It dates back some 50 years to Parker [1], though it does not appear to be widely known. It is not part of the standard calculus curriculum either at the secondary or tertiary level and I am yet to find the technique presented in any calculus text. This is surprising considering how simple the technique turns out to be.

As our first example using Parker's method, let us consider the evaluation of the integral

$$\int \frac{\sin^{-1}(x)}{x^2} dx.$$

By writing

$$G(x, f^{-1}(x)) = \frac{\sin^{-1}(x)}{x^2},$$

such that

$$y = f^{-1}(x) = \sin^{-1}(x) \quad \text{and} \quad x = f(y) = \sin(y),$$

then

$$G(f(y), y) = \frac{y}{\sin^2(y)}.$$

Upon applying equation (0.1) we have

$$\begin{aligned}
\int \frac{\sin^{-1}(x)}{x^2} dx &= \frac{y}{\sin^2(y)} \sin(y) - \int \sin(y) \frac{d}{dy} \left(\frac{y}{\sin^2(y)} \right) dy \\
&= \frac{y}{\sin(y)} - \int \sin(y) \left(\frac{\sin^2(y) - 2y \sin(y) \cos(y)}{\sin^4(y)} \right) dy \\
&= \frac{y}{\sin(y)} - \int \frac{\sin^2(y) - 2y \sin(y) \cos(y)}{\sin^3(y)} dy \\
&= \frac{y}{\sin(y)} - \int \text{cosec}(y)\, dy + 2 \int \frac{y \cos(y)}{\sin^2(y)} dy.
\end{aligned}$$

The integral of the cosecant is well known. The result is

$$\int \operatorname{cosec}(y)\, dy = -\ln(\operatorname{cosec}(y) + \cot(y)) = -\ln \left(\frac{1 + \cos(y)}{\sin(y)} \right).$$

In the second integral, if we put $x = \sin(y)$, so $dx = \cos(y) dy$, we see that

$$\int \frac{y \cos(y)}{\sin^2(y)} dy = \int \frac{\sin^{-1}(x)}{x^2} dx.$$

Thus

$$\int \frac{\sin^{-1}(x)}{x^2} dx = \frac{y}{\sin(y)} + \ln \left(\frac{1 + \cos(y)}{\sin(y)} \right) + 2 \int \frac{\sin^{-1}(x)}{x^2} dx.$$

Simplifying yields

$$\int \frac{\sin^{-1}(x)}{x^2}dx = -\frac{y}{\sin(y)} - \ln\left(\frac{1+\cos(y)}{\sin(y)}\right),$$

or in terms of x

$$\int \frac{\sin^{-1}(x)}{x^2}dx = -\frac{\sin^{-1}(x)}{x} - \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

As a second example using Parker's method let us consider the evaluation of the integral

$$\int (\sin^{-1}(x))^3\, dx.$$

Again, by writing

$$G(x, f^{-1}(x)) = (\sin^{-1}(x))^3,$$

such that

$$y = f^{-1}(x) = \sin^{-1}(x) \quad \text{and} \quad x = f(y) = \sin(y),$$

then

$$G(f(y), y) = y^3.$$

On applying equation (0.1) we have

$$\begin{aligned}
\int (\sin^{-1}(x))^3\, dx &= y^3\sin(y) - \int \sin(y)\frac{d}{dy}(y^3)dy \\
&= y^3\sin(y) - 3\int y^2\sin(y)dy \\
&= y^3\sin(y) + 3y^2\cos(y) - 6\int y\cos(y)\, dy \\
&= y^3\sin(y) + 3y^2\cos(y) - 6y\sin(y) + 6\int \sin(y)\, dy \\
&= y^3\sin(y) + 3y^2\cos(y) - 6y\sin(y) - 6\cos(y).
\end{aligned}$$

Here repeated integration by parts has been used. Finally, writing this in terms of x we have

$$\begin{aligned}
\int (\sin^{-1}(x))^3\, dx &= x(\sin^{-1}(x))^3 + 3(\sin^{-1}(x))^2\sqrt{1-x^2} \\
&\quad -6x\sin^{-1}(x) - 6\sqrt{1-x^2}.
\end{aligned}$$

As further practice in the application of Parker's method you may like to use it in order to show the following:

(i) $\int (\cos^{-1}(x))^2 dx = x(\cos^{-1}(x))^2 - 2\sqrt{1-x^2}\cos^{-1}(x) - 2x$

(ii) $\displaystyle\int x^2 \tan^{-1}(x)\, dx = \frac{x^3 \tan^{-1}(x)}{3} + \frac{\ln(x^2+1)}{6} - \frac{x^2}{6}$

(iii) $\displaystyle\int \frac{x^2 \tan^{-1}(x)}{1+x^2} dx = x \tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}[\tan^{-1}(x)]^2$

While the technique embodied in equation (0.1) for integrating more complicated expressions containing inverse functions is no doubt a powerful one, it is not without its caveats. As is the case with any technique of integration, deciding on when to use it is a skill in itself. In certain particular instances it turns out to be far easier to integrate such expressions directly by parts after the substitution of $y = f^{-1}(x)$ and any resulting algebraic simplifications are made. As an example, consider the integral

$$\int \frac{\sin^{-1}(x)}{\sqrt{(1-x^2)^3}} dx.$$

In this instance using Parker's method leads one into an almighty mess! Try it for yourself and see. If however one were to use the substitution of $y = f^{-1}(x) = \sin^{-1}(x)$, then $x = \sin y$ so that $dx = \cos y\, dy$. Upon substituting and simplifying the integrand, before integrating directly by parts, we have

$$\begin{aligned}
\int \frac{\sin^{-1}(x)}{\sqrt{(1-x^2)^3}} dx &= \int \frac{y \cos y}{\cos^3 y} dy \\
&= \int y \sec^2 y\, dy \\
&= y \tan y - \int \tan y\, dy \\
&= y \tan y + \ln(\cos y) \\
&= \frac{x \sin^{-1}(x)}{\sqrt{1-x^2}} + \frac{1}{2}\ln(1-x^2).
\end{aligned}$$

Reference

F. D. Parker, "Integrals of inverse functions," *American Mathematical Monthly* **62**, 439–440 (1955).