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The Nature of Mathematics

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Mathematics is a widely misunderstood subject. The common perception of mathematics is the subject filled with rules and procedures driven by very little motivation or meaning. Mathematics is often reduced to memorizing and replicating these procedures to solve endless identical problems, in which the procedure is promoted as the one way ticket to its solution. This is a tedious and frustrating experience for most, and even for those who succeed everything is usually forgotten after the big exam. Unfortunately most students do not get the chance to see that real mathematics is nothing like this.

Real mathematics is free from this meaningless grind, instead it is an endeavor that plays out much like a game. Real maths is driven by creativity and curiosity, it's definitely not something to be memorized. We can't squeeze all of maths into a giant textbook that we can look up every time we run into a problem: maths is not a completed subject set in stone, it's not even close and never will be.

To see the nature of mathematics, pretend that we are back at square one. We will start by creating the game of numbers. To make a game you need some things to play with, for instance we can imagine an object which we will call the number "1". Now it wouldn't be very exciting if this object sat around doing nothing, so we need some ways of playing with this new construct. The idea of addition is precisely an example of this, we take a pair of "1s" and use addition to combine them into a new object which we will call "2". We can symbolise this manipulation as "$1 + 1 = 2$". We can also add this new object to the old and we get $1+2 = 3$, producing another new object! We started off with one simple object, one manipulation and now we have the tools to generate as many new numbers as we please.

Let's pick up the pace. Instead of adding two numbers at a time, let's add many in one go:

$$2 + 2 + 2 + 2 + 2 + 2 = 12.$$

This is basically "6 lots of 2s". From this we feel the motivation to introduce a new way of manipulating numbers, the act of multiplication. This leads to the idea of

$$6 \times 2 = 12.$$

With the act of addition and multiplication and the ability to generate an infinite amount of numbers, we are now content with the depth of our game. So let's start exploring what we have just created.

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First we notice that $6 \times 2 = 12$ is not the only product that gives 12, as $3 \times 4 = 12$ and $1 \times 12 = 12$ are also true. We summarise this by saying that the factors of 12 are $1, 2, 3, 4, 6, 12$. However if we examine the number 11, we notice that $1 \times 11 = 11$ and there are no other ways of writing as a product of two numbers. Therefore 11 only has two factors 1 and 11, whereas 12 had many more. In this sense the number is special, and there other numbers like this, for example: 3, 5 and 7. These numbers deserve a special name: we call them the prime numbers.

Let's keep exploring the factors of numbers:

$$4 = 2 \times 2, 6 = 2 \times 3, 8 = 2 \times 2 \times 2, 9 = 3 \times 3, 10 = 2 \times 5, 14 = 2 \times 7.$$

At this point we see an odd pattern here, all these numbers can be written as a product of primes. But these numbers are selectively small, let's randomly jump to something larger, what about 3150?

$$3150 = 2 \times 5^2 \times 7 \times 9.$$

If you keep experimenting, you will find that writing any number as a product of primes is always possible. So naturally it is very tempting to guess that this is true for all numbers. So it seems like primes are the building blocks of all numbers, but there's more! If you investigate this further you will find that not only can all numbers be written as a product of primes, but this prime factorization is unique. For example, $170 = 2 \times 5 \times 17$ and you will not find any other set of prime numbers that multiply together to give 170. We can summarise our findings: *it appears that our game has the property that every number can be written as a product of prime numbers uniquely.*

We will pause here and reflect on what we have just achieved. We made a game with simple objects along with routine manipulations, yet from this seemingly predictable playing field we are able to extract an amazing property that was completely unexpected. We certainly didn't rig this property into the game during the creation process. Instead it turns out to be an intrinsic characteristic of our game that was discovered afterward as a result of our curiosity, our urge to explore what we have created. This is what drives mathematics. Mathematical objects never cease to surprise us with their endless elegant properties and structure. Although these properties are well hidden, all we need is a sense of curiosity to uncover them. Because of this, the ability to ask good questions is sometimes more valuable than being able to find answers.

Here is another odd pattern: it seems every even number can be expressed as a sum of two prime numbers. For example: $4 = 2 + 2,\ 6 = 3 + 3,\ 8 = 3 + 5,\ 10 = 3 + 7$, and so on.

Now I would like to highlight one aspect of mathematics that cannot be emphasised enough. This side of mathematics is absolutely essential, especially when dealing with new mathematical objects that are not simply numbers. This important aspect is our freedom to update our game of maths as we see fit, and we do this by adding new objects and new ways of manipulating these objects. Of course, with an updated game there will be more hidden properties to be found. Furthermore this update can be done by anyone, anywhere, anytime.

Now let's try updating our game of numbers. We have the positive numbers, but we have a strong urge to extend our game with the idea of negative numbers, as we want to introduce new objects into our game. But how does one introduce new numbers? Well one thing we have is a sense of how negative numbers should behave. For example, we can introduce the number -1 by saying -1 is the number which when added to 1 gives zero, that is $1 + (-1) = 0$. We can rinse and repeat this process to get all the negative numbers. Let's try multiplying these new negative numbers.

We could try to perform -2×3 but what should this equal to? An instinctive guess would be $-2 \times 3 = -6$. This guess is most likely inspired by our daily experiences. For example, -2 stands for owing someone two dollars. Now $\times \quad 3$ means we triple it, so we now owe six dollars, represented by -6. From this money analogy we are going to insist that is true, this is how our game of numbers is going to behave.

Now let's try -2×-3. We all know the answer is 6. But put yourself in the shoes of the first person to ever consider this product. You have never seen the answer $-2 \times -3 = 6$ and there are no textbooks to go to. So clearly at this point it is completely up to you to decide what -2×-3 produces. You are the designer of our number game.

$$-2 \times -3 = \textit{whatever you want}$$

We could choose some unrealistic answers, for example:

$$-2 \times -3 = 10 \textit{ million}$$
$$\text{or}$$
$$-2 \times -3 = \pi$$

It is within our freedom to do so. Tomorrow all the mathematicians could wake up and suddenly decide that $-2 \times -3 = \pi$ is the way to go and every single student who writes $-2 \times -3 = 6$ is wrong. Of course this would create utter chaos, because our number system would break down and the game of maths would just be nonsense and it would no longer be playable. We create our mathematical objects and the manipulations that go along with them, but we have to do this carefully in order to maintain a useful and consistent language of mathematics that is worth studying.

So if we want a working number system then we really only have two sensible options, namely:

$$\text{Option one:} -2 \times -3 = -6$$
$$\text{or}$$
$$\text{Option two:} -2 \times -3 = 6$$

Suppose we chose option one, but compare this to our earlier decision:

$$\text{From the money analogy:} -2 \times 3 = -6$$
$$\text{Option one:} -2 \times -3 = -6$$

And here we see something we don't like, that is, we take a certain number, -2 in this case, and multiply it by two very different numbers, and we get the exactly same result -6. And clearly this is not what we want, we expect multiplying by 3 and multiplying by -3 to give us different results, hence option two is the right choice by elimination.

Those who ventured far enough into mathematics would be aware of complex numbers, and this is just another example of an update, we are adding new objects into our game of numbers. However complex numbers are typically introduced as solutions to $x^2 + 1 = 0$, Not only is this uninspired, it is also historically incorrect. The real motivation leading up to the update of complex numbers and its foundations is much more interesting. While the updates to our game are for free, a truly worthy update to our game like the complex numbers is always motivated by genuine reasons and interesting results.

Not only can we bring new objects into our game, we also have the freedom to introduce new ways of manipulating them. So far we have addition and multiplication on numbers but to illustrate this freedom we will construct a brand new operation right now.

For instance, we could introduce an operation call "join" symbolised by $\otimes$, and this operation literally joins the numbers together:

$$2 \otimes 3 = 23$$
$$5 \otimes 10 = 510$$

We have just created a legitimate operation on numbers, but unfortunately we are not going to see this operation in textbooks tomorrow, it won't be heard in classrooms and it is not a worthy update to mathematics. This is because we did not show that our operator is capable of producing any interesting properties. However suppose we constructed another operator "$\sim$" which behaves such that:

$n \sim 3$ always seems to give us a prime number for any integer n or
$n \sim 2$ always seems to give us the number of factors of n

In this case we have interesting results that we can share, and this operator "$\sim$" will be treated very seriously and further investigation will open up and may even lead to brand new mathematics.

There is one more detail to be mentioned. We know $m + n = n + m$ for any numbers m and n. In technical language we say the addition operation is commutative. But notice that this is not true for our join operator $\otimes$. For example:

$$5 \otimes 10 = 510$$
$$10 \otimes 5 = 105$$

This is why the seemingly trivial commutative property is mentioned at all as not all operations are commutative so we need a piece of language to distinguish them.

Outside of numbers, algebra provides another place to see the creative side of mathematics. Let's begin by asking what is algebra? The typical response to this question is to give an example of algebra, something like:

$$\begin{aligned} y &= x + 1 \\ y &= 3x + 5 \\ x + 1 &= 3x + 5 \\ x &= -2 \end{aligned}$$

However this is not algebra, it is just routine manipulations. Real algebra is something a lot more interesting. In short, algebra is the study of algebraic structures, so let's examine what these are. An algebraic structure consists of two things: first a set of objects and these objects could literally be anything; secondly an operation that is able to combine any two objects from the set in some way to produce a third object from the same set. An obvious example of an algebraic structure is the set of positive integers $\{1, 2, 3, \ldots\}$ along with the addition operator $+$. Performing $1 + 7 = 8$ is an example of the operator combining two objects. However the basis of an algebraic structure is quite general and it captures many unexpected things. For example, consider the set of possible colours $\{green, blue, red, \ldots\}$ and the operation we are going to use will be the physical act of combining the colours. We can symbolize this operation with $*$, so $red * green = yellow$.

Problems involving symmetry also reduce down to algebra. If we rotate a square by $90°$ about its centre it looks the same (Figure 0), and we denote this rotational transformation as R_{90}. Other rotational symmetries of a square includes $R_{180}, R_{270}, R_{360}$. Aside

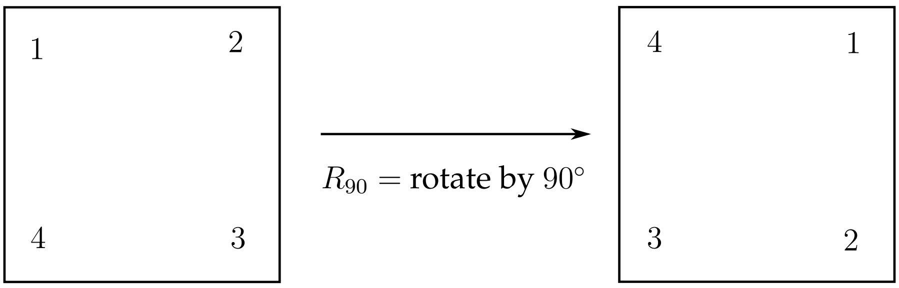

Figure 0: Rotational symmetry of a square

from spinning the square, we also see symmetry by flipping the square along certain lines (Figure 1). These flips are denoted as $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.

To link these transformations together we will use an operation $\circ$ that carries out the transformation in succession. So $R_{90} \circ \sigma_1$ means we first flip the square along the vertical then spin it by $90°$. We can verify that $R_{90} \circ \sigma_1 = \sigma_3$. If we gather all these transformations into a set

$$\textit{symmetry of a square} = \{R_{90}, R_{180}, R_{270}, R_{360}, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$$

along with the operator $\circ$ we now have an algebraic structure.

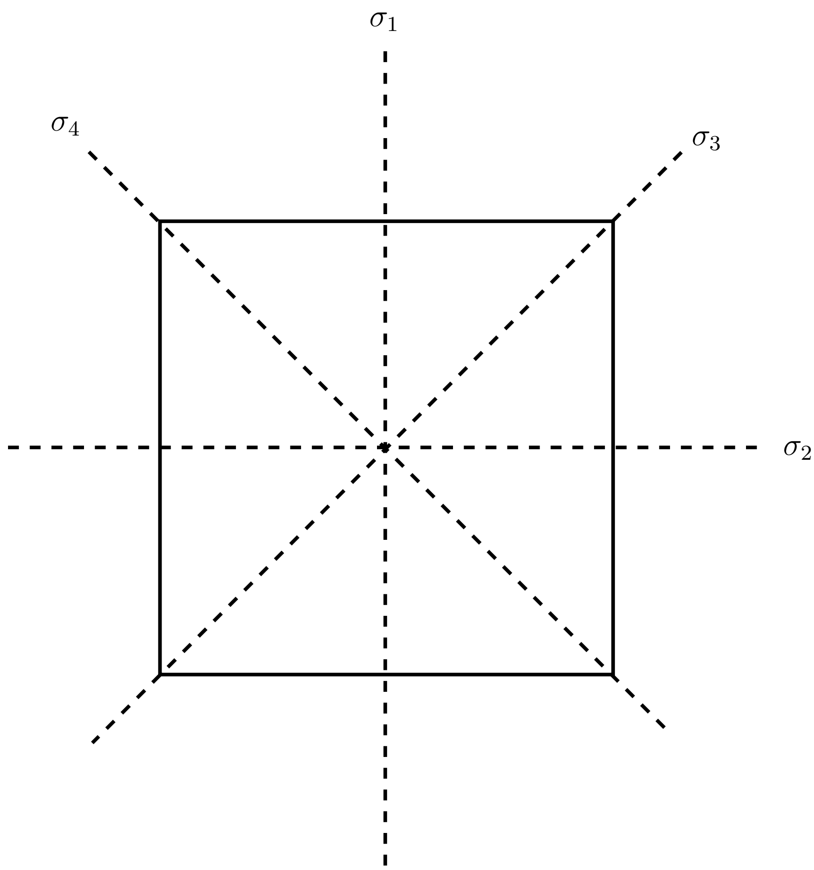

Figure 1: Reflective symmetries of a square

Various forms of communication such as the human voice, music, the internet, in fact all the possible sound and light waves we can perceive together with the act of superposition are other good examples of algebra. This list goes on and on: atomic structures, chemical reactions, functions, vectors, matrices etc. This highly generalizing aspect of algebra is what makes it so powerful.

With so many algebraic structures at hand, we should try to organise them into families based on how they behave. Here are some typical behaviours we can observe in algebraic structures:

- Some structures are commutative: the order in which the elements appear in an operation is irrelevant. For example, the set of numbers and $+$ forms a structure that is commutative as $m + n = n + m$ for any numbers m and n. The colours and the act of mixing them is also commutative. However rotating a square followed by a flip is different to first performing the flip then rotating. The symmetry structure of a square is not commutative.

- In some structures there seems to exist an identity element that has no effect during computations. For example 0 is the identity element for the additive number structure as $0 + n = n$ and $n + 0 = n$ for any number n. And 1 is the identity element for the multiplicative number structure as $1 \times n = n$ and $n \times 1 = n$ for any number. While there is no identity element in the colour structure, R_{360} is clearly the identity for the symmetry of a square.

- There are many other typical behaviours that I will leave for you to explore.

Next we choose some of these behaviours and use them as defining properties of a family. As a basic example we could examine the commutative family, that is, we collect all the algebraic structures that exhibit the commutative property. Now any properties we discover in regard to this family in general will automatically apply to all members of this family. This is a very powerful result because this gives us insight into unfamiliar territories and generates links between seemingly unrelated objects. Suppose we have a brand new abstract piece of mathematics but we are able to demonstrate that it belongs to a particular family of well-known structures. In this case absolutely everything we discovered about this structure automatically applies to the new. Ultimately we have established a deep connection, with this we can accelerate our understanding of the new piece of mathematics.

We have been playing games of numbers and algebra for thousands of years, but interestingly these games continue to reveal new intriguing properties despite their age. Now let us shift our attention to another game that has stood the test of time, the game of geometry. The basic objects here are points and lines, and we can combine and manipulate them into different shapes and study the properties that arise. Most would be familiar with this from their experiences in school, but the fact that there exist many different kinds of geometry is rarely addressed. An example is the ability to study geometry in abstract dimensions. Our real life experiences are trapped in three dimensional space but our imagination with mathematics is not. To enter this new geometry we need some tools, and the most basic of such is the idea of distance.

In one dimension (Figure 2) the distance between two points x_1 and x_2 is given by either $D_1 = x_2 - x_1$ or $D_1 = x_1 - x_2$ depending on which is bigger. However we can get rid of this ambiguity by saying $D_1 = \sqrt{(x_1 - x_2)^2}$.

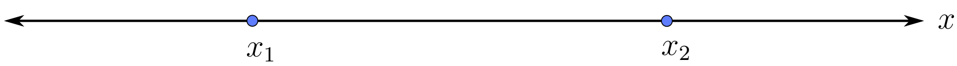

Figure 2: Two points in one dimensional space

Using Pythagoras' Theorem we know the distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in two dimensions (Figure 3) is given by:

$$D_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

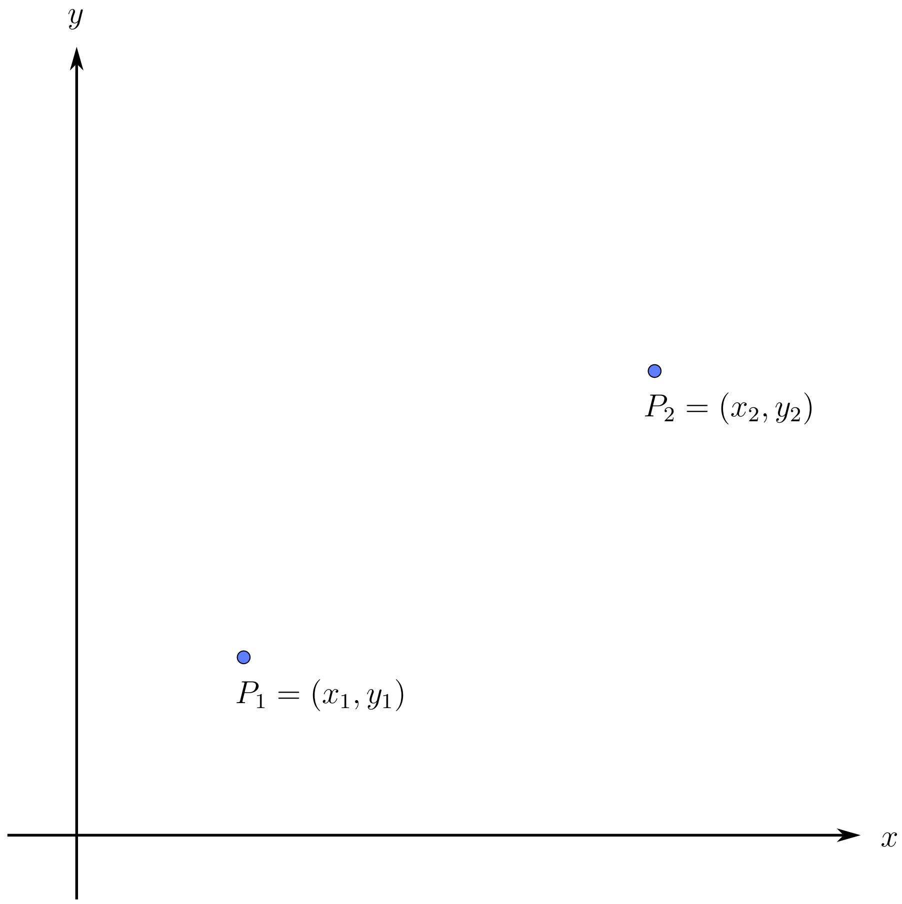

Figure 3: Two points in two-dimensional space

But what about the distance between two points in three dimensions (Figure 4)? If you do a bit of work (apply Pythagoras' Theorem twice) you will discover that:

$$D_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

And this result alone is interesting because the answer is so simple. We could have easily guessed this answer based on the pattern in the lower dimensions.

$$D_1 = \sqrt{(x_2 - x_1)^2}$$
$$D_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$D_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

But before the concrete proof is realised there is no reason why the answer could not have be something different and more complicated. The answer could have been

$$D_3 = \sqrt[3]{(x_2 - x_1)^3 + (y_2 - y_1)^3 + (z_2 - z_1)^3}$$

or even worse

$$D_3 = \frac{\sqrt[3]{(x_2 - x_1)^3 + (y_2 - y_1)^2 + (z_2 - z_1)^3}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

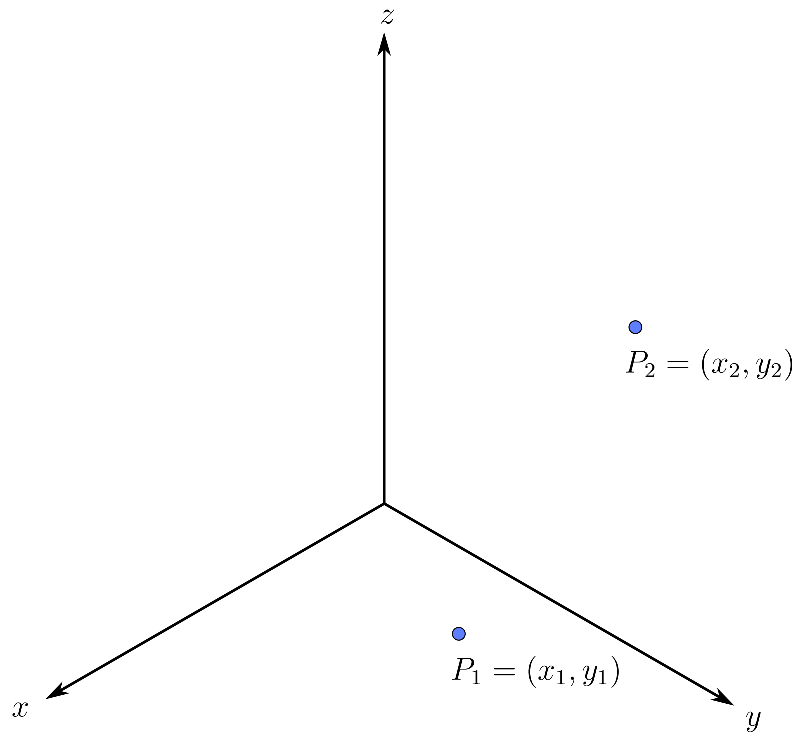

Figure 4: Two points in three-dimensional space

But the beautiful thing is that it's not. Instead the real answer follows a simple pattern. And these patterns show up time and time again in all areas of mathematics, so often that mathematicians have come to believe that this kind of elegance is something inherently built into mathematics. If any piece of mathematics is too long or tedious we can be certain that we are looking at it in the wrong way, and there exists a better approach hidden somewhere.

So now we have seen the idea of distances up to three spatial dimensions, and while we are bound to this in real life there is absolutely no reason why we should follow the same restrictions with mathematics. And getting to the idea of distances in abstract dimensions is easy, we simply follow the pattern! Since

$$D_1 = \sqrt{(x_2 - x_1)^2}$$

$$D_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

So we now define

$$D_4 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (\alpha_2 - \alpha_1)^2}$$

as the measure of distance between two points in four spatial dimensions.

It is clear that we can be stubborn and re-use this idea to define distances in any dimensions we like. We will probably never directly experience these abstract dimensions, we can't draw them and have no intuition for them, yet we can reach them mathematically in just a few simple steps. Overall this process is called generalisation and it is usually the key in extending many areas of mathematics.

Mathematicians have also updated regular geometry with new objects to create something new called projective geometry. To see the motivation behind this update we first examine regular geometry, where we are familiar with the two following facts:

1. Two points determine a line (Figure 5)

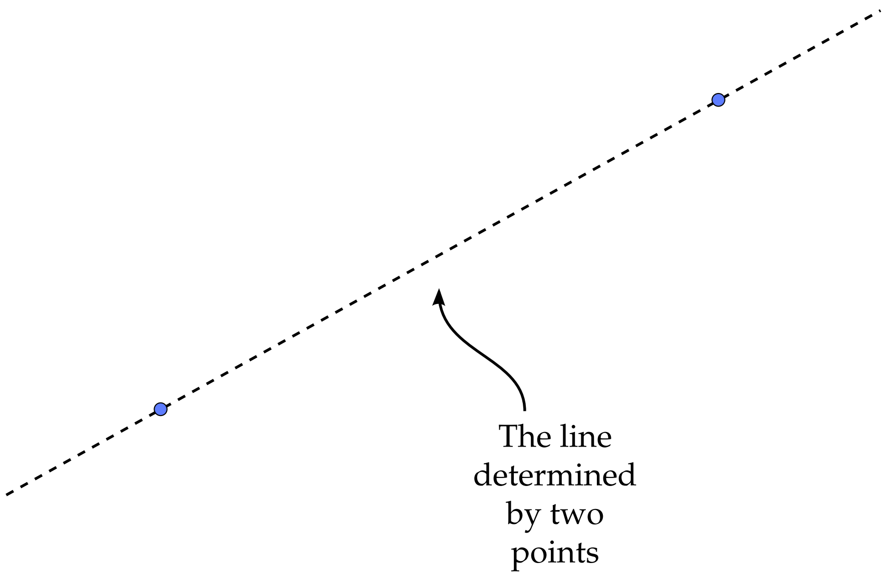

Figure 5: Two points determine a line

2. Two lines determine a point (Figure 6) except when two lines are parallel (Figure 7).

Mathematicians love elegance, and this exception is not elegant. To fix this problem mathematicians introduced a strange concept, ideal points. We give each family of parallel lines an ideal point where these lines can meet (Figure 8).

So with the inclusion of ideal points we now have:

- Two points determine a line.
- Two lines determine a point.

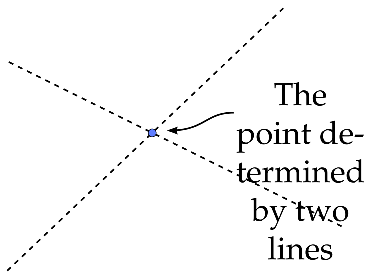

Figure 6

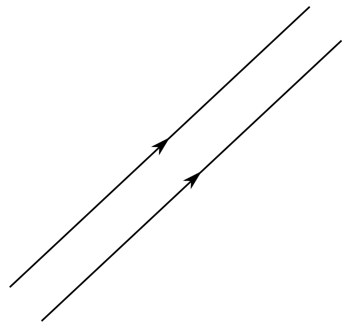

Figure 7

Ideal points give us a beautiful symmetry between lines and points but their existence seems questionable. This is because intuitively two parallel lines obviously never meet, no matter how far you look. However the purpose of ideal points is to give us a new geometry where parallel lines can meet in a new mathematical sense, something beyond our intuition. If we stop our discussion of ideal points here their existence will most likely remain unjustified. But instead of abandoning this seemingly bizarre idea we can choose to investigate further. In this case we will find some interesting results such as the ability to create a brand new operation on geometric objects called the cross product, also symbolised by "$\times$". The cross product of two lines l_1 and l_2, that is, the computation of $l_1 \times l_2$, gives us the point of the intersection of the two lines. The traditional method of finding such intersection involves rearranging the equations of the two lines in some arbitrary manner and solving a set of simultaneous equations. In projective geometry this tedious task of arbitrary substitutions and rearrangements is replaced by one straightforward systematic operation. Furthermore, we can also subject two points p_1 and p_2 to the cross product, and the computation $p_1 \times p_2$ will result in finding the line through the two points. So not only does symmetry exist in the geometric action between points and lines, it now also exists in their algebraic nature under this new geometry. In fact the distinctions between the roles of points and lines begin to disappear completely, making geometry simpler and more elegant. Needless to say, projective geometry has now earned a place in mathematics and it also plays a big role in arts and graphics. What started as an absurd idea turns out to have great merits.

The common strategy to make mathematics more interesting is to illustrate its applications in real life. A far more effective approach is to realise the imagination, the great insights and the elegance behind pure mathematics, an endeavour fascinating in its own right.

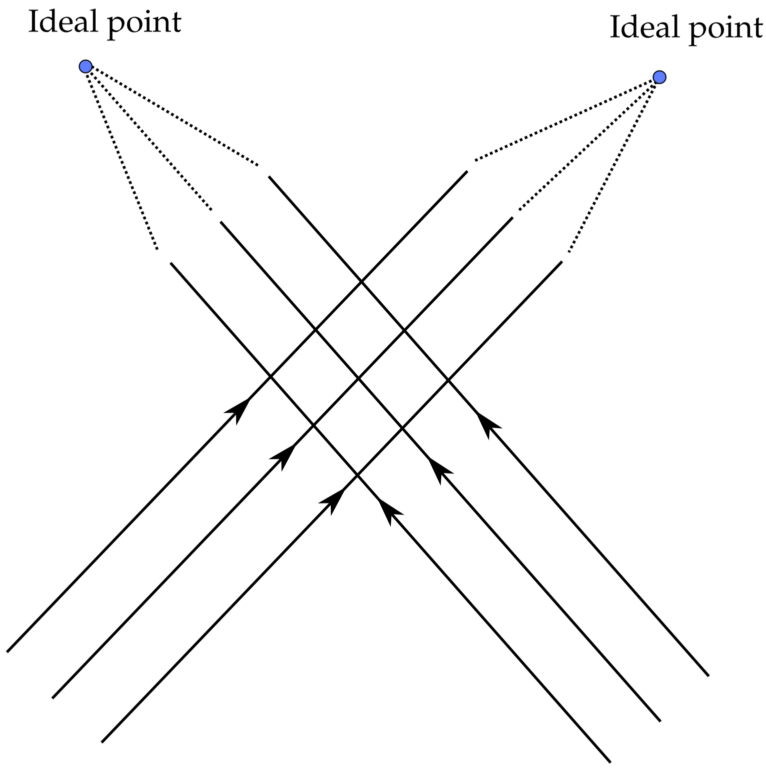

Figure 8