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Solutions 1371–1380

Q1371 Consider shuffles of a standard 52–card pack. (See the article in volume 47 number 3 for terminology and basic information.) Let rev be the shuffle which reverses the pack – that is, the first card is swapped with the last, the second with the second last, and so on. As in the article, out denotes the outshuffle.

(a) Write a formula for $rev(k)$ in terms of k; also, write rev as a product of cycles.

(b) Without any calculation, write down the cycle type of the composite shuffle $rev \circ out \circ rev$.

(c) Show that if we shuffle a pack of cards with both an outshuffle and a reverse shuffle, it makes no difference which one we do first.

SOLUTION

(a) Since $rev(1) = 52$, $rev(2) = 51$ and so on, it is not hard to see that

$$rev(k) = 53 - k \ .$$

To determine the cycles of rev we note that $rev(1) = 52$ and $rev(52) = 1$ and so on; therefore we just have a product of cycles with length 2, namely,

$$rev = (\,1\;\;52\,)(\,2\;\;51\,)(\,3\;\;50\,)\cdots(\,26\;\;27\,)\ .$$

(b) If the pack is reversed twice it returns to its original order: therefore the inverse of rev is just rev itself. So

$$rev \circ out \circ rev = rev^{-1} \circ out \circ rev\ ,$$

and this is a conjugate of out. Since cycle type is invariant under conjugation (previous issue, page 40), the cycle type of the given shuffle is the same as that of out: six cycles of length 8, one of length 2 and two of length 1.

(c) Recall from the article that the outshuffle is described by the formula

$$out(k) = \begin{cases} 2k-1 & \text{if } k = 1, 2, \ldots, 26 \\ 2k-52 & \text{if } k = 27, 28, \ldots, 52. \end{cases}$$

If $1 \le k \le 26$ then

$$rev(out(k)) = 53 - (2k-1) = 54 - 2k\ ;$$

and $27 \le rev(k) \le 52$, so

$$out(rev(k)) = 2(53-k) - 52 = 54 - 2k\ .$$

On the other hand, if $27 \le k \le 52$ then

$$rev(out(k)) = 53 - (2k - 52) = 105 - 2k$$

and

$$out(rev(k)) = 2(53 - k) - 1 = 105 - 2k\ .$$

So for all values of k we have $rev(out(k)) = out(rev(k))$, and we get the same result no matter which of the shuffles is performed first.

Q1372 Show that a composition of $n-1$ cycles

$$(\,1\,2\,) \circ (\,1\,3\,) \circ (\,1\,4\,) \circ \cdots \circ (\,1\,n\,)$$

can be written as a single cycle. Is

$$(\,1\,n\,) \circ \cdots \circ (\,1\,4\,) \circ (\,1\,3\,) \circ (\,1\,2\,)$$

the same cycle? For any numbers $a_1, a_2, \ldots, a_m$, describe the shuffle

$$(\,a_1\;a_2\;\cdots\;a_m\,) \circ (a_m\;\cdots\;a_2\;a_1\,)\ .$$

SOLUTION Remember that in a composition of cycles, the cycles are "active" from right to left. For the first problem, 1 is sent to n by the last pair and n is then unaffected by the other pairs; so 1 goes to n. Then n is sent to 1 by the last pair and to $n-1$ by the second last, so n goes to $n-1$. Proceeding similarly we see that the given expression is a single cycle

$$(\,1\;\;2\,) \circ (\,1\;\;3\,) \circ (\,1\;\;4\,) \circ \cdots \circ (\,1\;\;n\,) = (\,1\;\;n\;\;n{-}1\;\;\cdots\;\;2\,)\ .$$

In the same way we have

$$(\,1\;\;n\,) \circ \cdots \circ (\,1\;\;4\,) \circ (\,1\;\;3\,) \circ (\,1\;\;2\,) = (\,1\;\;2\;\;3\;\;\cdots\;\;n\,)$$

which is not the same. For the last question a_1 goes to a_m which then goes back to a_1, and likewise for all other elements. Therefore all elements remain unchanged by the given shuffle, and it is the identity shuffle.

Q1373 Find a natural cubic spline for the data points

$$(x, y) = (0, 4)\,,\ (1, 7)\,,\ (3, 5)\,.$$

(See the article in volume 47 number 3 for what this means and how to do it.)

SOLUTION Using the notation of volume 47 number 3, we need a pair of cubics

$$S_0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3$$
$$S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$$

with the properties

$$S_0(0) = 4 , \quad S_0(1) = 7 , \quad S_1(1) = 7 , \quad S_1(3) = 5 ,$$
$$S_0'(1) = S_1'(1) , \quad S_0''(1) = S_1''(1) , \quad S_0''(0) = 0 , \quad S_1''(3) = 0 .$$

Working out the required derivatives and making the necessary substitutions gives

$$\begin{gathered} a_0 = 4 , \quad a_0 + b_0 + c_0 + d_0 = 7 , \\ a_1 = 7 , \quad a_1 + 2b_1 + 4c_1 + 8d_1 = 5 , \\ b_0 + 2c_0 + 3d_0 = b_1 , \quad 2c_0 + 6d_0 = 2c_1 , \\ 2c_0 = 0 , \quad 2c_1 + 12d_1 = 0 . \end{gathered}$$

We have immediately $a_0 = 4$ and $c_0 = 0$ and $a_1 = 7$; substituting into the other equations and making obvious simplifications gives

$$b_0 + d_0 = 3 \tag{0.1}$$
$$b_1 + 2c_1 + 4d_1 = -1 \tag{0.2}$$
$$b_0 + 3d_0 = b_1 \tag{0.3}$$
$$3d_0 = c_1 \tag{0.4}$$
$$c_1 + 6d_1 = 0 \tag{0.5}$$

To solve these we write b_1, c_1, d_1 in terms of d_0: equations (1) and (3) give $b_1 = 2d_0 + 3$, while (4) and (5) give $d_0 + 2d_1 = 0$. Substituting back into (2) yields

$$(2d_0 + 3) + 2(3d_0) + 2(-d_0) = -1 ;$$

hence $d_0 = -\frac{2}{3}$ and it is then easy to find $b_0 = \frac{11}{3}$, $b_1 = \frac{5}{3}$, $c_1 = -2$ and $d_1 = \frac{1}{3}$. So our spline is given by

$$f(x) = \begin{cases} 4 + \frac{11}{3}x - \frac{2}{3}x^3 & \text{if } 0 \le x \le 1 \\ 7 + \frac{5}{3}(x-1) - 2(x-1)^2 + \frac{1}{3}(x-1)^3 & \text{if } 1 \le x \le 3. \end{cases}$$

Q1374 Postman Pat has an infinite number of letters to deliver in Integer Avenue, which contains an infinite number of houses. Each letter is labelled with an integer (positive, negative or zero), and different letters have different labels. The letter with label n is to be delivered to house number $(4n+1)^2$. Prove that no house receives more than one letter.

SOLUTION Suppose that two different letters, labelled m and n, are both delivered to the same house. Then $(4m+1)^2 = (4n+1)^2$. This means that either

$$4m + 1 = 4n + 1 , \quad \text{so} \quad m = n ,$$

or

$$4m + 1 = -(4n + 1) , \quad \text{so} \quad 4m + 4n = -2 .$$

But both of these are impossible, the first because the letters have different labels, the second because the left-hand side is a multiple of 4 and the right-hand side is not. So two letters cannot be delivered to the same house.

Q1375 Find a triangle with integer side lengths, all different, such that the longest side is 2011 and one of the angles is $\cos^{-1}\frac{1102}{2011}$.

SOLUTION Try a triangle $\triangle ABC$ with $AB = 2011$ and $AM = 1102$, where M is the point on AC such that BM is perpendicular to AC. This clearly has the required dimensions; since $BM^2 = 2011^2 - 1102^2 = 2829717$ we need to find $y = BC$ and $x = MC$ such that

$$y^2 - x^2 = 2829717\ ,$$

that is,

$$(y+x)(y-x) = 3^2 \times 11 \times 101 \times 283\ .$$

In order to make x and y as small as possible we need to write the right-hand side as a product of two factors which are as close together as possible, which means that they must be as close as possible to $\sqrt{2829717} \approx 1682$. A bit of trial and error shows that the best choice is

$$y + x = 3^2 \times 283 = 2547 \quad \text{and} \quad y - x = 11 \times 101 = 1111\ ;$$

this leads to $y = 1829$ and $x = 718$, giving a triangle with sides 2011, 1829 and 1820.

Correct solution received from John Barton, Victoria.

Q1376 Prove that any number of the form $n = 17272\ldots27271$ is a multiple of 19.

SOLUTION Adding $10n$ to n, we have

$$\begin{aligned} 11n &= 17272\ldots272710 + 17272\ldots27271 \\ &= 18999\ldots99981 \\ &= 19000\ldots00000 - 19 \end{aligned}$$

which is obviously a multiple of 19. Since 11 and 19 have no common factor, n itself is also a multiple of 19. If you are not familiar with the result used in the previous sentence you can do it this way: since $11n$ is a multiple of 19, so is $77n$; and $76n$ is a multiple of 19 (because $76 = 4 \times 19$); so the difference $77n - 76n$ is a multiple of 19. That is, n is a multiple of 19.

Correct solution received from John Barton, Victoria.

Q1377 Let a be a positive integer. Prove that if n is an integer and $n > 2a^2$, then $n! > a^n$.

SOLUTION Observe that $n!$ is given by a product of n numbers,

$$n! = n(n-1)(n-2)\cdots(a^2+1)(a^2)(a^2-1)\cdots(3)(2)(1)\ ,$$

in which $n - a^2$ of the factors are greater than a^2. Therefore

$$n! > (a^2)^{n-a^2} = a^{2n-2a^2} = a^n a^{n-2a^2}\ ,$$

and if $n > 2a^2$ then we have $n! > a^n$ as required.

Q1378 Suppose that x satisfies the equation

$$x^{10^{10^{10}}} = 10^{10^{10^{11}}} .$$

We would like to know how many decimal digits x has.

(a) What would you guess? Ten digits? A hundred? A million maybe?

(b) Calculate the actual answer.

SOLUTION

(a) Up to you!

(b) We have

$$x = 10^{10^{10^{11}}/10^{10^{10}}} = 10^{10^{10^{11}-10^{10}}} = 10^{10^{90000000000}}$$

and so the number of digits in x is $1 + 10^{90000000000}$. Rather more than a million, which is 10^6. Note that the number just given is not x itself, but the number of digits in x.

For problems related to this, see `www.gbbservices.com/math/large.html`.

Correct solution received from John Barton, Victoria.

Q1379 If

$$\begin{aligned} 123x + 456y + 789z &= a \\ 456x + 789y + 123z &= b \\ 789x + 123y + 456z &= c\,, \end{aligned}$$

find *with minimum possible hard work* x, y, z in terms of a, b, c.

SOLUTION Adding all three equations gives

$$1368x + 1368y + 1368z = a + b + c\ ,$$

while subtracting the first from the second gives

$$333x + 333y - 666z = b - a\ .$$

So

$$3z = (x + y + z) - (x + y - 2z) = \frac{a + b + c}{1368} - \frac{b - a}{333}$$

This simplifies to

$$z = \frac{189a - 115b + 37c}{151848}\ ,$$

and in the same way we get

$$x = \frac{189c - 115a + 37b}{151848}\ , \quad y = \frac{189b - 115c + 37a}{151848}\ .$$

Correct solution received from John Barton, Victoria.

Q1380 I form a list of numbers by sticking together two consecutive positive integers. Thus I start with $12, 23, 34$; a bit later I get $89, 910, 1011, 1112$; I keep on going until I reach $9798, 9899, 99100$. What is the sum of all these numbers?

SOLUTION The "second parts" of these numbers are $2, 3, 4, \ldots, 100$. Using the formula for the sum of an arithmetic progression, they add up to $\frac{1}{2}(99)(102) = 5049$. The "first parts" are

$$10,\ 20,\ 30,\ \ldots,\ 80,\ 900,\ 1000,\ 1100,\ \ldots,\ 9700,\ 9800,\ 99000$$

which add up to $\frac{1}{2}(8)(90) + \frac{1}{2}(90)(10700) + 99000$. The grand total is 585909.

Solutions to earlier problems were received from John Barton (problems 1362, 1363, 1364, 1365, 1366, 1369, 13700; and slso from David Shaw, Geelong (problem 1368).