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Solving Second Order Recurrences

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First consider a **sequence**, by which we mean a list of numbers which goes on forever. An example is the well-known *Fibonacci sequence*

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ \ldots$$

in which every number (except for the first two) is the sum of the previous two. If we refer to these numbers, using function notation, as $a(0), a(1), a(2), a(3)$ and so on, the sequence is specified by the **recurrence relation**

$$a(n) = a(n-1) + a(n-2) \quad \text{for} \quad n \geq 2 \tag{1}$$

together with the **initial conditions**

$$a(0) \quad \text{and} \quad a(1) = 1\ .$$

Equation (1) is referred to as a **second order** recurrence because each number is determined by the previous two numbers. If we wish, we can easily make up a third or higher order recurrence: for example, we might decide to write down a list in which each number is equal to the previous number, plus 14 times the one before that, minus 24 times the previous one again. That is,

$$a(n) = a(n-1) + 14a(n-2) - 24a(n-3) \quad \text{for} \quad n \geq 3\ .$$

In this case we would have to specify three initial conditions, for example,

$$a(0) = 1 \quad \text{and} \quad a(1) = 2 \quad \text{and} \quad a(2) = 1\ ,$$

giving the sequence

$$1,\ 2,\ 1,\ 5,\ -29,\ 17,\ -509,\ 425,\ -7109,\ 11057,\ -98669,\ \ldots\ .$$

Many counting problems can be modelled by recurrence relations. As an example we give an application of mathematics to the important(?) topic of interior design. Suppose that a 2 metre wide hallway is to be covered with carpet tiles. Two kinds of tiles are available: a 2 metre by 2 metre square, and a 2 metre by 1 metre rectangle. It is not possible to cut the tiles or to overlap them. If the hallway is n metres long, how many design options are there? OK, not exactly a question which is going to change

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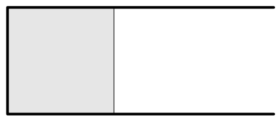
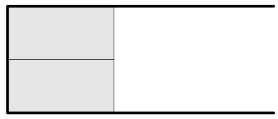
the course of civilisation, but it will illustrate how to approach this kind of problem. The first step is to define appropriate notation: let $a(n)$ be the number of design options if the hallway is n metres long. Next, we shall set up a recurrence relation for $a(n)$ by considering how we can begin the carpeting at one end of the hallway. A little thought indicates that there are three ways to cover the beginning of the hallway, as shown in the diagrams.





If we begin with a square, as in the first diagram, then the task of finishing the carpeting of the n metre hallway is exactly the same as that of carpeting an $n - 2$ metre hallway. The number of ways to do this is unknown; but the point is that it is essentially the *same* unknown as in our original problem, the only difference being the length of the hallway. So the number of ways is $a(n - 2)$. Similarly, if we begin as in the second diagram, we then need, in effect, to carpet an $n - 1$ metre hallway, and the number of ways to do so is $a(n-1)$. In the third diagram once again there are $a(n-2)$ ways to finish the job. Therefore the total number of ways to carpet the hallway is $a(n-1) + 2a(n-2)$. On the other hand, we have defined the number of ways to be $a(n)$, and so we have

$$a(n) = a(n - 1) + 2a(n - 2) \ . \tag{2}$$

This clearly makes no sense if $n = 1$ or $n = 2$, so we have to consider these cases separately to give the initial conditions

$$a(1) = 1 \quad \text{and} \quad a(2) = 3 \ . \tag{3}$$

These conditions enable us to determine all values of $a(n)$; for example, by doing a lot of step–by–step calculation we could find $a(1000)$, the number of ways of carpeting a 1 kilometre long(!) hallway. Often, however, we would like to find a formula for $a(n)$ directly in terms of n. This is referred to as **solving the recurrence**, and is the main topic of this article.

So, let's solve the recurrence relation (2), given the initial conditions (3). First we shall determine numbers r and s for which (2) can be rewritten

$$[a(n) - ra(n - 1)] = s[a(n - 1) - ra(n - 2)] \ . \tag{4}$$

To do so, expand (4) and rearrange to give

$$a(n) = (r+s)a(n-1) - rsa(n-2)\ ; \tag{5}$$

if this is to be the same as (2) we need

$$r+s=1 \quad \text{and} \quad rs=-2\ .$$

These equations can be solved in various ways, for example, by noting that r and s must be solutions of the quadratic $x^2 - x - 2 = 0$. We find that $r = -1$ and $s = 2$, or possibly the other way around: it doesn't actually matter which. Therefore we have written (2) as

$$[a(n) + a(n-1)] = 2[a(n-1) + a(n-2)]\ .$$

Now remember that n is not a fixed number but a variable; the equation we have just written is still true if n is replaced by $n-1$ or $n-2$ and so on. Therefore

$$\begin{aligned} a(n) + a(n-1) &= 2[a(n-1) + a(n-2)] \\ &= 2^2[a(n-2) + a(n-3)] \\ &= 2^3[a(n-3) + a(n-4)] \\ &= \cdots \\ &= 2^{n-2}[a(2) + a(1)]\ ; \end{aligned}$$

but we know that $a(2) + a(1) = 4$, and so

$$a(n) + a(n-1) = 2^n\ .$$

This equation also is true for all values of n and we can write

$$\begin{aligned} a(n) + a(n-1) &= 2^n \\ a(n-1) + a(n-2) &= 2^{n-1} \\ a(n-2) + a(n-3) &= 2^{n-2} \\ &\ \vdots \\ a(2) + a(1) &= 2^2. \end{aligned}$$

Now take the first of these equations, minus the second, plus the third and so on. Most of the terms on the left-hand side drop out and we obtain

$$a(n) + (-1)^n a(1) = 2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-2}2^2\ .$$

Finally, we know that $a(1) = 1$; and the right-hand side can be added up as a geometric series; a little algebra gives the explicit formula

$$a(n) = \frac{2^{n+1} + (-1)^n}{3}\ .$$

For example, we can calculate $a(10) = 683$: there are 683 ways to carpet a 10-metre hallway with the available resources. In fact, this kind of problem is usually solved by a slightly simpler method, essentially the same but hiding a lot of the work. But you cannot understand the working if it's hidden from you! So the above is not a bad way to start: maybe you will learn the "advanced" method later. We illustrate again by finding a "direct" formula for the Fibonacci numbers.

We begin in the same way, expanding (4) to give (5), and equate (5) with the Fibonacci recurrence (1). We need $r + s = 1$, $rs = -1$, and with a bit of care we obtain the values

$$r = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad s = \frac{1-\sqrt{5}}{2}.$$

As mentioned above, it does not matter if we interchange r and s. Now we have

$$\begin{aligned} a(n) - ra(n-1) &= s[a(n-1) - ra(n-2)] \\ &= s^2[a(n-2) - ra(n-3)] \\ &= \cdots \\ &= s^{n-1}[a(1) - ra(0)] \\ &= s^{n-1}. \end{aligned}$$

Observe that we have gone one step further than last time, because in this problem $a(0)$ is meaningful whereas in the previous one it was not. At this point a little more work is needed: we shall replace n by $n-1$, $n-2$, and so on, and each time we shall multiply the equation by r. This gives

$$\begin{aligned} a(n) - ra(n-1) &= s^{n-1} \\ ra(n-1) - r^2a(n-2) &= rs^{n-2} \\ r^2a(n-2) - r^3a(n-3) &= r^2s^{n-3} \\ &\vdots \\ r^{n-2}a(2) - r^{n-1}a(1) &= r^{n-2}s \\ r^{n-1}a(1) - r^na(0) &= r^{n-1}. \end{aligned}$$

If we add all these equations we see once again that pretty much everything disappears on the left-hand side,

$$a(n) - r^na(0) = s^{n-1} + rs^{n-2} + r^2s^{n-3} + \cdots + r^{n-2}s + r^{n-1} .$$

Now $a(0) = 0$; and the right-hand side is a geometric series with ratio r/s; doing the algebra gives

$$a(n) = \frac{r^n - s^n}{r - s} .$$

Substituting in the known values of r and s, we obtain

$$a(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) .$$

This is a pretty amazing formula—since we already know that $a(n)$ is an integer, what are all those $\sqrt{5}$ terms doing there?? In fact, if you use the Binomial Theorem to expand the expression, say for $n = 5$, you will find that all of the square roots cancel (not to mention the fractions) and the final answer is indeed an integer.

If you study the method given you should find that you can solve virtually any second order recurrence (though there are occasional complications). You might even try to devise a similar method for a third order recurrence!

Exercise. In both of these problems we mentioned that it does not matter if we interchange the values of r and s. Go back to the first problem, take $r = 2$, $s = -1$ and work through the same ideas; you should find that the calculations are slightly different but the final answer is identical.