Parabola Volume 48, Issue 2 (2012)

Not Enough Monkeys . . .

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A popular classical problem can be stated as follows:

> There are five monks, one monkey and pile of coconuts on a desert island. One monk goes to the pile of coconuts, gives one to the monkey, removes a fifth of the remaining coconuts, buries them and goes to sleep.
>
> The second monk then wakes up, goes to the pile of coconuts, gives one to the monkey, buries a fifth of what remains and goes to sleep. The other monks do likewise. In due course all five monks wake up and go to the pile of coconuts which they then succeed in sharing equally among them.
>
> What is the smallest possible number of coconuts that the pile originally contained?
>
> If after the final division there is still one coconut left for the monkey, what is the smallest possible number in the original pile?

We consider an island with n monks and k monkeys, denote

$$k \equiv j \mod (n)$$

whenever n divides $k - j$ exactly, and define the integer $\sigma_n(k)$ by

- $\sigma_n(k) \equiv -k \mod (n)$
- $1 \leq \sigma_n(k) \leq n$

i.e.

$$\sigma_n(k) = n - k + n \left\lfloor \frac{k}{n} \right\rfloor$$

where $\left\lfloor \frac{k}{n} \right\rfloor$ is the largest integer smaller or equal to $\frac{k}{n}$.

We have

- $\sigma_n(k) = \sigma_n(j)$ only if $k \equiv j \mod (n)$
- $\sigma_n(k) + \sigma_n(j) = n$ only if $k + j \equiv 0 \mod (n)$

hence

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- $\sigma_n(k^{2i-1}) = n - \sigma_n\left((\sigma_n(k))^{2i-1}\right),\ i = 1, 2, 3, \ldots$
- $\sigma_n(k^{2i}) = \sigma_n\left((\sigma_n(k))^{2i}\right),\ i = 1, 2, 3, \ldots.$

Now, each monk goes to the pile of coconuts, gives one to each monkey then buries a fraction $\frac{k}{k+\lambda}$, $\lambda \geq 1$, of what's remaining.

Thus the number of coconuts $x_{n,k}^{(i)}(\lambda, r)$ left in the pile by the ith monk, given that after the final division there are still r coconuts left, is related to $x_{n,k}^{(i-1)}(\lambda, r)$ by

$$x_{n,k}^{(i)}(\lambda, r) = \frac{\lambda}{k+\lambda}\left(x_{n,k}^{(i-1)}(\lambda, r) - k\right), \quad i = 1, 2, \ldots, n$$

leading to

$$x_{n,k}^{(i)}(\lambda, r) = c_n(\lambda, r)\lambda^i(k+\lambda)^{n-i} - \lambda$$

where $c_n(\lambda, r)$ is a positive integer, independent of k since $x_{n,k}^{(n)}(\lambda, r)$ is.

The size of the original pile is

$$x_{n,k}(\lambda, r) = x_{n,k}^{(0)}(\lambda, r) = c_n(\lambda, r)(k+\lambda)^n - \lambda$$

and each monk's share of coconuts after the final division is

$$\begin{aligned} y_n(\lambda, r) &= \frac{1}{n}\left(x_{n,k}^{(n)}(\lambda, r) - r\right) \\ &= \frac{1}{n}\left(c_n(\lambda, r)\lambda^n - \lambda - r\right). \end{aligned}$$

Thus $c_n(\lambda, r)$ is given by

$$c_n(\lambda, r)\lambda^n \equiv \lambda + r \mod (n)$$

and is restricted to the interval $[1, n]$ for the smallest solution $x_{n,k}(\lambda, r)$.

If n is prime then k^{-1}, the reciprocal of k, defined by

$$kk^{-1} \equiv 1 \mod (n)$$

is such that

$$\begin{aligned} \sigma_n(k^{-1}) &= (\sigma_n(k))^{-1} \\ k^{-1} &= \sigma_n\left((\sigma_n(k))^{n-2}\right) \end{aligned}$$

since, by Euler, $k^n \equiv k \mod (n)$.

We have

$$\begin{aligned} c_n(\lambda, r) &\equiv \lambda^{-n}(\lambda + r) \mod (n) \\ &\equiv \lambda^{-1}(\lambda + r) \mod (n) \\ &\equiv 1 + r\lambda^{-1} \mod (n) \\ &\equiv 1 - r\left(\sigma_n(\lambda)\right)^{-1} \mod (n) \end{aligned}$$

hence

$$c_n(\lambda, r) = \sigma_n \left(r \left(\sigma_n(\lambda) \right)^{-1} - 1 \right).$$

Thus on an island with 5 monks and 1 monkey, where each monk buries a fraction $\frac{1}{5}$ of the current pile, the first monk ought to start from a pile of size

$$x_{5,1}(4, r) = 3125\, \sigma_5(r - 1) - 4$$

in order to end up, as well as his colleagues, with a yield of

$$y_5(4, r) = \frac{1}{5} \left(1024\, \sigma_5(r - 1) - r - 4 \right)$$

and r coconuts remaining, namely $x_{5,1}(4, 0) = 3121$, $y_5(4, 0) = 204$ and $x_{5,1}(4, 1) = 15621$, $y_5(4, 1) = 1023$.

For the same r, this yield remains the same with $k \geq 2$ monkeys and burying a fraction of $\frac{k}{k+4}$ by each monk. More monkeys please!!!