Parabola Volume 48, Issue 2 (2012)

Solutions 1381–1390

Q1381 It is commonly believed that the minute hand and the hour hand on a clock are in *exactly* symmetrical positions when the time is $10:08$ and 42 seconds.

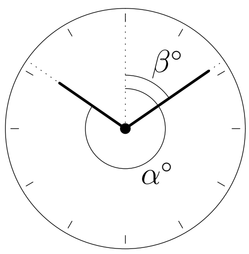
(a) Without detailed calculations, prove that this is wrong.

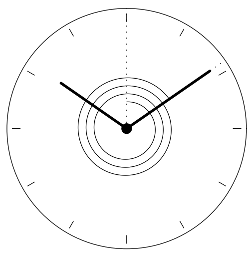
(b) When, at about this time, are the hands exactly symmetrical?

SOLUTION

(a) Consider the minute markings on the clock face. At $10:08:42$ the minute hand is between the eighth and ninth markings on the right-hand side. The hour hand travels at one-twelfth the speed of the minute hand, and so it takes 12 minutes to travel from one mark to the next. At $10:00$ it is at the tenth marking down on the left-hand side; at $10:12$ it is at the ninth marking; so at $10:08:42$ it is between the ninth and tenth markings, and it is not symmetrical with the minute hand.

(b) Suppose that the hands are in a symmetric position when the hour hand has travelled an angle α degrees from the vertical; let the angle between the minute hand and the vertical be β degrees (see the first diagram). Because of symmetry, $\beta = 360 - \alpha$. To reach its present position the minute hand has actually travelled right round the clock face, say k times (see the second diagram, pictured with $k = 3$) and it has moved through an angle of $\beta + 360k$ degrees.





Since the minute hand travels at 12 times the speed of the hour hand we have

$$\beta + 360k = 12\alpha\ ;$$

substituting $\beta = 360 - \alpha$ and solving gives

$$\alpha = \frac{k+1}{13}\,360\ .$$

The case we want is just after 10 o'clock, so $k = 10$. Since the hour hand travels $360°$ in 12 hours, the time it has taken is

$$\frac{11}{13} \times 360 \times \frac{12}{360}\ \text{hours} \quad = \quad \frac{132}{13}\ \text{hours}$$

$$
\begin{aligned}
&= 10 \text{ hours } \frac{120}{13} \text{ minutes}\\
&= 10 \text{ hours } 9 \text{ minutes } \frac{180}{13} \text{ seconds}\\
&= 10 \text{ hours } 9 \text{ minutes } 13\tfrac{11}{13} \text{ seconds}\ .
\end{aligned}
$$

Therefore the hands are exactly symmetric at $10:09:13\frac{11}{13}$.

NOW TRY problem 1391.

Q1382 Which is bigger: the ratio of the volume of a sphere to that of its inscribed cube, or the ratio of the volume of a cube to that of its inscribed sphere?

SOLUTION For a sphere of radius r with an inscribed cube of side length s, the diagonal of the cube is a diameter of the sphere and so $s\sqrt{3} = 2r$; so the ratio of volumes is

$$\frac{\frac{4}{3}\pi r^3}{s^3} = \frac{4\pi r^3}{3}\frac{3\sqrt{3}}{8r^3} = \frac{\pi\sqrt{3}}{2}\ .$$

For a cube of side length s with an inscribed sphere of radius r the diameter of the sphere is equal to the side of the cube, $2r = s$, and the ratio of volumes is

$$\frac{s^3}{\frac{4}{3}\pi r^3} = \frac{2s^3}{\frac{1}{3}\pi s^3} = \frac{6}{\pi}\ .$$

You can evaluate these two quantities by calculator (boring), or you can make some simple comparisons

$$\frac{\pi\sqrt{3}}{2} > \frac{3 \times 1{\cdot}5}{2} > 2 \quad \text{and} \quad \frac{6}{\pi} < \frac{6}{3} = 2$$

to see that the ratio of sphere to inscribed cube is the larger.

Q1383 Choose as many as possible of the integers $1, 2, 3, \ldots, 100$, subject to the following conditions:

- Every pair of your chosen numbers must have a common factor greater than 1.
- There must be no number greater than 1 which is a factor of *all* your chosen numbers.

SOLUTION (This is only a partial solution: let us know if you can improve on it!) Consider the set of numbers

$$
\begin{aligned}
\{\, &6, 10, 12, 15, 18, 20, 24, 30, 36, 40, 42, 45, 48,\\
&\qquad 50, 54, 60, 66, 70, 72, 75, 78, 80, 84, 90, 96, 100 \,\}
\end{aligned}
$$

which consists of all numbers from 1 to 100 which are multiples of 6 or of 10 or of 15. A multiple of 6 and a multiple of 10 have a common factor of 2 (maybe more); a multiple of 6 and a multiple of 15 have a common factor of 3; a multiple of 10 and a multiple of

15 have a common factor of 5. Thus any pair of numbers from the set has a common factor greater than 1. On the other hand there is no number other than 1 which is a factor of 6 and 10 and 15, and so certainly none which is a factor of every element in the set. Therefore it is possible to choose 26 numbers satisfying the stated conditions; I do not know whether or not it is possible to get more than 26.

Q1384 For any real number x we write $\lfloor x \rfloor$ for x rounded to the nearest integer downwards, and $\lceil x \rceil$ for x rounded to the nearest integer upwards. For example,

$$\lfloor \pi \rfloor = 3 \quad \text{and} \quad \lceil \pi \rceil = 4 \quad \text{and} \quad \lfloor 5 \rfloor = \lceil 5 \rceil = 5 \; .$$

Find all real numbers x which satisfy the equation

$$\lfloor 6x \rfloor + 7x - \lceil 8x \rceil = 9 \; .$$

SOLUTION Let $a = \lfloor 6x \rfloor$ and $b = \lceil 8x \rceil$ and $\alpha = 6x - a$ and $\beta = b - 8x$. Then a, b are integers and $0 \le \alpha, \beta < 1$. Notice that

$$7x = 9 - a + b \; , \tag{1}$$

so $7x$ is an integer, and

$$\alpha - \beta = 14x - a - b \; ,$$

so $\alpha - \beta$ is also an integer. As α and β both lie between 0 and 1, the only way their difference can be an integer is if they are equal. Thus we have

$$6x = a + \alpha \; , \quad 8x = b - \alpha \; , \tag{2}$$

where a, b are integers and $0 \le \alpha < 1$. Now we can eliminate x and b from equations (1) and (2) to give

$$5a = 54 + 7\alpha \; .$$

However 7α is less than 7, so this means that $5a$ is a multiple of 5 lying between 54 and 61. There are two possibilities, $a = 11, 12$; and then from previous equations it is easy to find the respective values of α and x. We have two potential solutions

$$x = \frac{13}{7} \quad \text{and} \quad x = \frac{15}{7} \; ,$$

and substituting back confirms that these both work.

NOW TRY problem 1392.

Q1385 An n-digit integer is a string of n decimal digits with no prohibition on leading zeros: for example, 005105 is a valid six-digit integer. How many triples (x, y, z) of 10-digit integers are there which satisfy the conditions that x, y and z contain the digits $0, 1$ and 5 only, and that $x + y = z$?

SOLUTION Let $a(n)$ be the number of triples of n-digit integers satisfying the required conditions. Consider the pairs of corresponding digits in x and y. For example, the six-digit solution

$$150050 + 001051 = 151101$$

corresponds to the pairs

$$(1,0)\,,\ (5,0)\,,\ (0,1)\,,\ (0,0)\,,\ (5,5)\,,\ (0,1)\,.$$

The pairs $(1,1)$, $(1,5)$ and $(5,1)$ cannot occur as each would give a digit of 2 or 6 in z (or a 3 or 7 if there is a "carry" from the following digit). The pair $(0,1)$ can occur but must not be followed by $(5,5)$ since then the sum $0+1$, plus a carry of 1 from the following column, would give a digit 2 in z; the same is true for the pairs $(1,0)$, $(0,5)$ and $(5,0)$. The first pair cannot be $(5,5)$ as then z would have an extra digit.

Consider numbers having at least two digits, and first consider the case when the second pair of digits is not $(5,5)$. Then our n-digit numbers contain one of five possibilities for the first pair, followed by $n-1$ pairs which form an acceptable pair of $(n-1)$-digit numbers. So the number of solutions in this case is

$$5a(n-1)\,.$$

Next consider the case when the second pair is $(5,5)$. If the third pair is not $(5,5)$ then the last $n-2$ pairs form acceptable numbers; if it is but the fourth is not, then the last $n-3$ pairs form acceptable numbers; and so on. Therefore the number of solutions in this case is

$$a(n-2) + a(n-3) + \cdots + a(1) + 1\,,$$

where the final 1 is accounting for the possibility that all pairs except the first are $(5,5)$. So the total number of solutions, when $n \geq 2$, is given by

$$a(n) = 5a(n-1) + a(n-2) + a(n-3) + \cdots + a(1) + 1\,; \qquad (*)$$

it is easy to see that $a(1) = 5$, so

$$a(2) = 5 \times 5 + 1 = 26\,, \quad a(3) = 5 \times 26 + 5 + 1 = 136,$$

and so on. In fact we can make things a bit easier for ourselves by observing that if $n \geq 3$ then $(*)$ holds both for n and for $n-1$, so

$$\begin{aligned} a(n) &= 5a(n-1) + a(n-2) + a(n-3) + \cdots + a(1) + 1 \\ a(n-1) &= 5a(n-2) + a(n-3) + a(n-4) + \cdots + a(1) + 1\,. \end{aligned}$$

Subtracting these equations and rearranging gives

$$a(n) = 6a(n-1) - 4a(n-2) \quad \text{for} \quad n \geq 3\,.$$

So we have

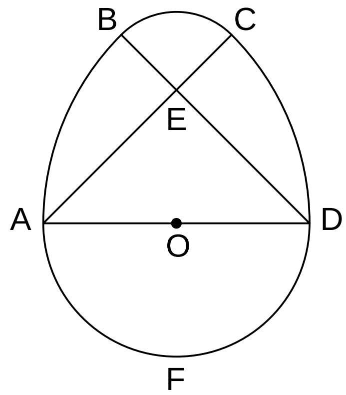
$$a(1) = 5$$

$$\begin{aligned} a(2) &= 26 \\ a(3) &= 6\times 26 - 4\times 5 \;=\; 136 \\ a(4) &= 6\times 136 - 4\times 26 \;=\; 712 \end{aligned}$$

and so on, which eventually gives our answer of $a(10) = 14\,672\,384$.

NOW TRY problem 1393.

Q1386 An egg shape is constructed as in the diagram. The line segment AD has length 2, and O is its midpoint. The angles $\angle CAD$ and $\angle BDA$ are each 45°. The curves AB, BC and CD are circular arcs with centres D, E, A respectively, and AFD is a semicircle with centre O. Find the area of the egg.



SOLUTION The egg consists of

- a semicircular region $AODF$ with radius 1;
- two 45° circular sectors CAD and BDA with radius $AD = 2$, which overlap in a right-angled isosceles triangle with hypotenuse 2;
- a quarter-circle region EBC with radius $2-\sqrt{2}$.

Therefore the total area is

$$\frac{\pi}{2} + 2\,\frac{4\pi}{8} - \frac{\sqrt{2}\sqrt{2}}{2} + \frac{\pi(2-\sqrt{2})^2}{4} = \bigl(3-\sqrt{2}\bigr)\pi - 1\ .$$

Q1387 Find all solutions of the simultaneous equations in $2n$ variables

$$\begin{array}{cc} x_1^2 + x_2 = 1\ , & 2x_2 + x_3 = 1 \\ x_3^2 + x_4 = 1\ , & 2x_4 + x_5 = 1 \\ \vdots & \vdots \\ x_{2n-1}^2 + x_{2n} = 1\ , & 2x_{2n} + x_1 = 1\ . \end{array}$$

SOLUTION First assume that $|x_1| \le 1$. Then we can write $x_1 = \cos\theta$ and we have

$$x_2 = 1 - \cos^2\theta = \sin^2\theta\ , \quad x_3 = 1 - 2\sin^2\theta = \cos 2\theta\ .$$

It is easy to show inductively that

$$x_{2k} = \sin^2(2^{k-1}\theta) \quad \text{and} \quad x_{2k+1} = \cos(2^k\theta) \ , \tag{1}$$

and we therefore have a solution provided that

$$2\sin^2(2^{n-1}\theta) + \cos\theta = 1 \ ,$$

that is, $\cos\theta = \cos(2^n\theta)$. The solutions

$$\theta = \frac{2m\pi}{2^n - 1} \quad \text{for} \quad m = 0, 1, 2, \ldots, 2^{n-1} - 1 \tag{2a}$$

and

$$\theta = \frac{2m\pi}{2^n + 1} \quad \text{for} \quad m = 1, 2, 3, \ldots, 2^{n-1} \tag{2b}$$

are all different and between 0 and π, and therefore give 2^n different values for $x_1 = \cos\theta$.

On the other hand, by elementary algebra we have

$$x_2 = 1 - x_1^2 \ , \quad x_3 = -1 + 2x_1^2 \ , \quad x_4 = 4x_1^2 - 4x_1^4$$

and we can show that x_{2k} is a polynomial of degree 2^k in x_1. So the equation $2x_{2n} + x_1 = 1$ becomes a polynomial of degree 2^n in x_1; since we have already found 2^n possible values of x_1, there are no more to find! Therefore all solutions of the system are given by the formulae (1) with the values of θ given by $(2a)$ and $(2b)$.

Alternative solution Adding 16 times all the quadratic equations, subtracting 8 times the others, and completing squares, gives

$$(4x_1 - 1)^2 + (4x_3 - 1)^2 + \cdots + (4x_{2n-1} - 1)^2 = 9n \ .$$

So there is at least one k for which

$$(4x_{2k-1} - 1)^2 \leq 9$$

and hence $-1 \leq x_{2k-1} \leq 1$. But then it is easy to see from the original equations that x_{2k-3} satisfies the same inequality; eventually we have $-1 \leq x_1 \leq 1$ and we proceed as in the previous solution.

Q1388 Consider the sequence of numbers obtained by stringing together the digits of the positive integers, namely

$$1 \ , \ 12 \ , \ 123 \ , \ 1234 \ , \ 12345 \ , \ 123456 \ , \ldots$$
$$\ldots , \ 12345678910 \ , \ 1234567891011 \ , \ 123456789101112$$

and so on. Prove that three consecutive numbers in this sequence can never have any common factor (except for 1).

SOLUTION The key observation is that any common factor of a and $aq + b$ is also a factor of b. Let m_1, m_2, m_3 be three consecutive numbers in the list; then m_2 and m_3 are obtained by appending the digits of integers (say) n and $n + 1$, respectively, to the previous numbers, and so

$$m_2 = 10^{d_1} m_1 + n \ , \quad m_3 = 10^{d_2} m_2 + (n + 1)$$

for some exponents d_1, d_2. From our "key observation", any number which is a common factor of m_1, m_2 and m_3 must be a factor of both n and $n + 1$; using the same observation again, the common factor must be a factor of 1, and therefore can only be equal to 1.

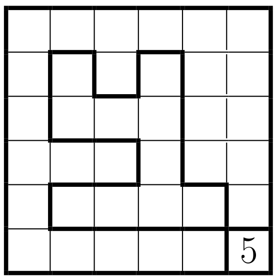
Q1389 (inspired by the popular KenKen® puzzle (`www.kenken.com`)) Fill in one of the numbers $1, 2, 3, 4, 5, 6$ in each empty square of the diagram given below, in accordance with the following rules.

(a) Each horizontal row must contain the numbers 1 to 6, once each.

(b) Each vertical column must contain the numbers 1 to 6, once each.

(c) The numbers in the "inner" region of ten squares must have a sum of 26.

(d) The numbers in the "outer" region of twenty–five squares must have a product of 24186470400000.



SOLUTION First we note that $24186470400000 = 2^{17} \times 3^{10} \times 5^5$. Therefore the numbers in the "outer" region must include five 5s; the sixth is obviously in the bottom-right square. Suppose that the numbers of 6s, 4s, 3s, 2s and 1s in the outer region are a, b, c, d, e respectively. Now this region contains twenty-five numbers (including the five 5s), so

$$a + b + c + d + e = 20 \ .$$

Also, the total of all thirty-six numbers is $6(1 + 2 + 3 + 4 + 5 + 6) = 126$, so the total of those in the outer region is $126 - 26 - 5 = 95$. Once again, this includes the five 5s, so

$$6a + 4b + 3c + 2d + e = 70 \ .$$

The numbers in the outer region must have altogether seventeen factors of 2 and ten factors of 3, so

$$a + 2b + d = 17 \quad \text{and} \quad a + c = 10 \ .$$

Eliminating a, b, c from these four equations gives

$$d + 3e = 9\ . \tag{$*$}$$

Now look at the bottom row and the right-hand column. Each of these must contain a 6, a 4, a 3, a 2 and a 1; so $a, b, c, d, e \geq 2$. Therefore $(*)$ gives only one possibility $e = 2$, $d = 3$, and then it is not hard to find $c = 6$, $b = 5$, $a = 4$. Now we know exactly what numbers are in each region of the diagram, and it remains to arrange them correctly.

Consider which numbers go around the edges, and which go in the "indents" of the outer region. There are six 3s and five 4s; only four of each can go around the edges, so the indents must contain two 3s and a 4. Because of the 5 in the bottom right-hand corner, only two more 5s can go around the edges, and the other three must occupy the indents. It is not hard to see that they must be arranged as follows.

		5		3	
				5	
	5	3		4	
					5

Now we know that the central region contains no 5s or 3s, so it is easy to place the rest of these numbers. Also, the central region contains four 1s, and so the two 1s in the outer region must occupy two of the three corners; clearly they must be in the top-right and bottom-left corners. So we get

	3		5		1
		5		3	
3				5	
	5	3		4	
5					3
1			3		5

The bottom row and right-hand column must each contain a 6, a 4 and a 2; this leaves two 6s, two 4s and a 2 in the top row and left-hand column. Clearly the 2 must be in the corner, and it is now not very difficult to fill in the rest of the puzzle.

2	3	4	5	6	1
4	1	5	2	3	6
3	2	1	6	5	4
6	5	3	1	4	2
5	6	2	4	1	3
1	4	6	3	2	5

Q1390 I have a bag of counters labelled with the numbers 1 to 2012, once each. I am to choose counters from the bag, but I am not allowed to take any two counters whose numbers differ by 10. What is the maximum number of counters I can take?

SOLUTION We begin by dividing the numbers $\{1, 2, \ldots, 2012\}$ into pairs as follows:

$$\begin{gathered} \{1, 11\},\ \{2, 12\}, \ldots,\ \{10, 20\}, \\ \{21, 31\},\ \{22, 32\}, \ldots,\ \{30, 40\}, \\ \cdots \\ \{1981, 1991\},\ \{1982, 1992\}, \ldots,\ \{1990, 2000\}. \end{gathered}$$

At this point the pattern changes as we are about to run out of numbers. At some stage we'll have to switch over to single numbers,

$$\{2001, 2011\},\ \{2002, 2012\},\ \{2003\},\ \{2004\}, \ldots,\ \{2010\}.$$

We have written down 1010 sets, and I can only take one number from each set. So I cannot get more than 1010 numbers. And I can actually get this many by choosing the smaller number in every set. Therefore the maximum number of counters I can take is 1010.