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History of Mathematics: Winning strategies

Michael A B Deakin[1]

This column's topic will be a theorem in the mathematical theory of games. It concerns what could be thought of the simplest possible type of game we can imagine.

> It is played by 2 players who move alternately.
>
> It is a game of perfect information; there are no hidden data such as would occur with a card or dice game; both players are fully aware at all times of the state of the game.
>
> The game ends within a finite number of moves.
>
> It is impossible to have a drawn game; one or the other of the players must win.

For definiteness, call the players Alice and Bob.[2] Alice is supposed to have the first move and Bob the reply and so the game proceeds, with the two taking turn and turn about until one or other of them wins. For such a game, there is a (at first sight rather surprising) theorem:

> Either Alice can force a win, whatever Bob does (in other words, Alice has a "winning strategy"),
>
> or else Bob can force a win, whatever Alice does (in other words, Bob has a "winning strategy").

It is a little difficult to give a precise origin to this theorem. The name of the German Ernst Zermelo (1871–1953) is now attached to it because of a 1913 paper he wrote, making especial reference to the game of chess.[3] Another name that is often invoked in this connection is that of the French mathematician Émile Borel (1871–1956), who is credited by some as, in essence, deriving the same result in a series of papers written in the 1920s. The first really clear and explicit statement was however given by a Polish

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[1]Dr Michael Deakin is an Adjunct Senior Research Fellow in the School of Mathematical Sciences at Monash University.

[2]These names, if not exactly standard, have somehow become widely used in discussions such as this.

[3]The theorem, as stated above, does not apply to chess, where draws can occur. It can however be extended to cover such cases. I leave this extension to the reader to explore. In the case of chess, it is widely believed that with perfect play on both sides, every game should end in a draw, in other words that one only loses in chess by making a mistake. This belief has not been, and probably never will be, rigorously proved!

mathematician, Hugo Steinhaus (1887–1972), who published it in Polish and in a very obscure journal in 1925. However, his paper has since been translated into English by the US Navy and published in their *Logistics Quarterly* (1960). Steinhaus's proof is by far the best, and is the one I will give here.

However, the result may well predate all these writers (as some websites intimate); it is the sort of thing that can easily be accepted in the absence of formal statement.

To follow Steinhaus's proof, we need a few preliminaries drawn from mathematical logic. The symbol $\forall$ is a shorthand way of saying "for all"; the symbol $\exists$ is to be read as "there exists"; the symbol $\neg$ represents "not"; S is a shorthand for an assertion or "statement", which is asserted to be true. A full stop may be read as "for which" or "such that", but in many cases, we do not need these extra words in ordinary English. Thus the string of symbols $\forall x.S$ reads as "for all values of x, statement S is true", i.e. statement S holds true whatever value x has; the string of symbols $\exists x.S$ reads as "there exists a value of x for which statement S holds true". The statement $\neg S$ is to be interpreted as "S is false".

We now come to De Morgan's two laws of mathematical logic. They are named after the mathematician Augustus De Morgan (1806–1871) and go as follows.

$$\neg\forall x.S \equiv \exists x.\neg S$$

$$\neg\exists x.S \equiv \forall x.\neg S.$$

In other words, the symbol $\neg$ turns a subsequent $\forall$ into a $\exists$ and vice versa. Translating these laws into ordinary English, we can see that they are really obvious. The first goes: "To say that it is not true that S holds for all x is the same thing as saying that there is some x for which S is false". The second is similar: "To say that it is not true that there is an x for which S is true is the same thing as saying that for every x, S is false".

These laws also apply to composite statements, and indeed this is where the benefit of the mathematical shorthand really asserts itself. So for example $\neg\forall x.\exists y.S \equiv \exists x.\neg\exists y.S \equiv \exists x.\forall y.\neg S$. This is to write (much more succinctly): "To say that it is not true that for all x there is a y such that S holds is the same thing as saying that there exists an x such that for all y, S is false".

We are now ready to apply these laws to the proof of the winning strategy theorem. Let S be the statement "Alice wins the game"; let T be the statement: "Alice has a winning strategy". Then T amounts to saying that Alice has a move such that whatever reply Bob makes, she will have a move such that whatever reply Bob makes, she will have a move ...... Alice wins. Put this into mathematical formalism with obvious notation:

$$T \equiv \exists a_1.\forall b_1.\exists a_2.\forall b_2.\exists a_3.\forall b_3 .... \ S.$$

Now T must be either true or false, so the only other possibility is $\neg T$. But we can write this, using De Morgan's laws over and over again as:

$$\neg T \equiv \forall a_1.\exists b_1.\forall a_2.\exists b_2.\forall a_3.\exists b_3 \ .... \ \neg S.$$

"Whatever Alice does, Bob has a reply such that whatever Alice does .... Alice doesn't win, i.e. Alice loses and Bob wins."

Now it is important to realize the restrictions the theorem envisages. It rules out most card games and also dice games such as Backgammon. Nor does it apply without modification to chess, for which draws are possible.

However, there are many games that *do* come under the theorem's aegis. A very familiar one is Nim, in which several piles of objects are presented and players alternate in removing objects from one pile or another until one or other player picks up the final object.

Nim comes in many different versions, all of which have been completely analyzed mathematically. Most mathematical accounts of Nim concentrate on the "forward game" versions for which the winner is the player who picks up the last of the objects under some rule or other.

When Nim is played in social settings such as bar-rooms, etc., it is usually played as a *misère* game in which the player who picks up the last object loses. The winning strategies for these versions are thus different from those pertaining to the "forward game", although the general lines of the analysis are much the same.[4]

If you Google "game of nim", you will find many good accounts of the various versions that have been proposed. Essentially, the number of piles and the number of objects in each together determine which player should win.

Another such game is Hex. This was invented independently by Piet Hein (of Grooks fame) and John Nash (of *A Beautiful Mind*).

This too comes in various versions. One of these was marketed by the firm *Parker Brothers* and it was they who provided the name. Unlike Nim, Hex has eluded complete mathematical analysis. Nash proved that draws are impossible in Hex, and he was also able to show by means of a subtle and ingenious argument that Alice has a winning strategy. It is not known, however, what that winning strategy is, except in some of the simplest cases.

Again a Google search for "game of hex" will supply a lot of detail.

For the rest of this article, however, I will concentrate on a much less well known game. I first learned of it from an account given to an international conference in 1988 by the French mathematician Claude Berge (1926–2002), an eminent authority on the theory of games. Berge called the game **Northeast**, but it is also known by other names. It was described explicitly in a short note in the *American Mathematical Monthly* (1974, pp. 876–879). This was written by the American mathematician David Gale, also a specialist in the theory of games. Reading between the lines, I surmise that Gale himself may not have invented it, but rather brought it to public attention. Nonetheless he suggested the name **Gnim** for it, the G perhaps standing for "Gale"! However, before his article saw print, Gale sent an advance copy to the *Scientific American's* mathematical columnist Martin Gardner, who called it **Chomp**, and included a discussion of it in his column for January 1973, with further comments continuing until May of that year. The game was later discussed by Berge in a review paper on such games in 1982, and

[4]The anayses all use binary arithmetic.

in that context he called it **Gnim**. In the subsequent 6 years, he must have adopted the alternative name **Northeast.**[5]

It is not easy to find material on Northeast on the internet (or anywhere else for that matter), but after much searching, I finally discovered a website which (*inter alia*) gives a quite good account. Go to

http://www.econ.ohio-state.edu/jpeck/Econ601/Econ601L10.pdf

Alternatively you can visit

http://en.wikipedia.org/wiki/Chomp[6]

These give readable and accurate summaries. However, here I will use a slightly different (although equivalent) description. Begin with a rectangular pattern of objects, shown below as asterisks. The dimensions are taken to be $m \times n$, which is to say that there are m (horizontal) rows and n (vertical) columns. (Here $m = 6$ and $n = 8$.)

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*  *  *  *  *  *  *  *
*  *  *  *  *  *  *  *
*  *  *  *  *  *  *  *
*  *  *  *  *  *  *  *
*  *  *  *  *  *  *  *
*  *  *  *  *  *  *  *
```

Our two players, Alice and Bob, set out to play, with Alice moving first. To describe the moves, I will use a sort of coordinate system. The asterisk in the ith column from the left and the jth row from the bottom will be designated (i, j). Alice chooses one of the asterisks and removes it from the pattern along with all those above it ("north"), to its right ("east"), or both ("northeast"). In the diagram below, I show the effect of choosing (4, 3). I have replaced this asterisk by a barely visible dot, and completely removed all those others.

```
*  *  *
*  *  *
*  *  *
*  *  *  ·
*  *  *  *  *  *  *  *
*  *  *  *  *  *  *  *
```

[5]Nim, Hex and Northeast are all classified as *impartial games*, which means that any legal play is available to whichever player has the move. Contrast chess, where one player may only move the white pieces and the other the black. It should be noted however that the winning strategy theorem applies whether or not a game is impartial.

[6]Both websites accessed 19/9/2011.

Bob must now remove one of the remaining asterisks together with all those to its "north", "east"and "northeast". The game progresses until one or other of the players is forced to choose the asterisk at the bottom left, that with coordinates (1, 1), the "poisoned asterisk". The player removing this asterisk loses the game.[7]

It is quite easy to prove that Alice has a winning strategy. Here is how the proof goes.[8]

The asterisk at (n, m) is the one farthest from the one to be avoided. Taking this leads either to a forced win or a forced loss, by the theorem stated and proved above. If it is a winning move, we need look no further. If it is a losing move, then it loses because Bob, in response, can choose a winning move (i, j). But it would have been open to Alice to make this move instead of (n, m). In that case, Alice would win.[9]

The interesting point, however, is that in many cases, we don't know how to advise Alice to proceed. We know *that* she has a winning strategy, but we don't know *what* that strategy is. Of course, in any particular case, with enough patience, we can determine what Alice is to do.

Start with the case $m = 1$. Here the winning move is $(2, 1)$. Anything else loses, and so for this case, as long as $n > 2$, the strategy of choosing the "farthest" asterisk fails, because Bob in reply could play $(2, 1)$ and so force Alice to take the poisoned asterisk. A similar analysis applies to the case $n = 1$.

Now let us look at another very simple case: $m = n = 2$. In this instance, the "farthest" asterisk is $(2, 2)$. Suppose that Alice takes this. It is now very easy to see that Bob is a goner. If he takes $(1, 2)$, Alice replies with $(2, 1)$ and if he takes $(2, 1)$ Alice replies with $(1, 2)$, and either way Bob is left with the poisoned asterisk.

This case is actually the key to one of the few general results known, that where $m = n$ and the rectangle is square. Here Alice's winning move is $(2, 2)$. This removes all the asterisks except for the poisoned one and those lying on two "arms" emanating from it to the "north" and the "east". Bob is thus forced to take some asterisks from one of these, and all Alice has to do to preserve the win is to take an equal number from the other. The win comes about because Alice can always restore the equality of the lengths of the "arms". Note however that this only works for squares. Were Alice to try this strategy for any other rectangle, she would lose because then it would be Bob who could bring about the equality.

It is instructive to note that $(2, 2)$ is the *only* move Alice can make to preserve her winning advantage. Had she taken any asterisk north, east or northeast of $(2, 2)$, then Bob could reply $(2, 2)$ and so win. On the other hand, if Alice had taken an asterisk with either i or j equal to 1 (that is to say, on one of the "arms"), then she would have left a rectangle with (in essence) Bob having first move, and we know that in such a circumstance *he* has a winning strategy (although we do not know in general what it is).

Now that the case of a square pattern has been disposed of, we can turn our atten-

[7]Thus Northeast is classified as a *misère* game.

[8]I am here neglecting the trivial case $n = k = 1$. This stands apart, because in this instance, Bob wins (by default, without even making a move)!

[9]This proof depends upon the fact that Northeast is an impartial game.

tion to the more general case. We saw in the very simple early situations that the cases $m = 1$ and $n = 1$ were equivalent. The same holds true for any rectangular pattern. So we lose no generality by assuming that $m < n$.

Look now at the case $m = 2$, i.e. that in which there are just 2 rows. Start with the case $n = 3$. Here the "farthest asterisk" approach works for Alice. She plays $(3, 2)$. If now Bob chooses $(3, 1)$, then he leaves Alice with the move in the 2×2 situation, and so with a win. If he plays $(2, 1)$, Alice plays $(1, 2)$ and wins. Finally if Bob plays $(2, 2)$, Alice plays $(3, 1)$, to set up a winning configuration we have seen before.

This illustrates the more general case. In the $2 \times n$ case, Alice chooses $(n, 2)$, and if Bob chooses $(i, 1)$ for some i, he will leave a perfectly rectangular pattern with Alice to move and so retain a winning advantage. In this case, she will play $(i - 1, 2)$, and proceed from there. On the other hand, if Bob chooses $(i, 2)$ for some i, then Alice plays $(i + 1, 1)$, and the pattern confronting Bob is much as before. Alice systematically reduces the size of the array until it becomes obvious that Bob has no chance of winning the game.

Because we can now assume that $m < n$ and lose no generality thereby, the next case in order of difficulty is 3×4. This is already quite involved. Here it is.

$$\begin{array}{ccccc} 3 & * & * & * & * \\ 2 & * & * & * & * \\ 1 & * & * & * & * \\ & 1 & 2 & 3 & 4 \end{array}$$

To discover Alice's winning strategy, start with the plays she cannot make. $(1, 1)$ is obviously out; $(2, 1)$, $(3, 1)$, $(4, 1)$, $(1, 2)$ and $(1, 3)$ all lead to a win for Bob, because he has the move when the pattern is rectangular; $(2, 2)$ is also out, as we have already seen. This leaves 5 possible plays to be considered: $(3, 2)$, $(4, 2)$, $(2, 3)$, $(3, 3)$ and $(4, 3)$.

Consider first $(3, 2)$, producing the pattern shown here:

$$\begin{array}{ccccc} 3 & * & * & & \\ 2 & * & * & & \\ 1 & * & * & * & * \\ & 1 & 2 & 3 & 4 \end{array}$$

Bob now has eight possible replies:

$(1, 1)$ is suicide

$(2, 1)$ loses immediately to $(1, 2)$

$(3, 1)$ sets up a rectangular pattern, and so Alice wins (following the 2×3 pattern already discussed, but this time extending upwards rather than to the right)

$(4, 1)$ loses to $(1, 3)$, again reaching a position already discussed

$(1, 2)$ loses immediately to $(2, 1)$

$(2,2)$ loses to $(4,1)$, once again reaching a familiar position

$(1,3)$ also loses to $(4,1)$, yet again reaching a familiar position

$(2,3)$ loses to $(3,1)$ and here also we see a sequence of plays encountered before, but again extending upwards rather than to the right.

Thus Alice can win by choosing $(3,2)$. Interestingly, if she plays anything else, she loses. $(4,2)$, $(3,3)$ and $(4,3)$ all allow Bob to play $(3,2)$ and so to win. On the other hand, if she plays $(2,3)$, Bob can reply $(3,1)$ and so reach a winning position, another that we've seen before.

This case shows how complicated the game can become in very short order. Perhaps it is not so surprising that a general rule for the winning strategy is not known! However, this has not prevented further investigation. Gale looked further at the cases $m = 3$ and found the following winning moves:

$$n = 5,\ (4,3);\quad n = 6,\ (4,2);\quad n = 7,\ (5,3);\quad n = 8,\ (6,2);$$
$$n = 9,\ (7,3);\quad n = 10,\ (6,2);\quad n = 11,\ (7,2);\quad n = 12,\ (9,3).$$

He continued a computer analysis up to $n = 100$ and noted that in each case, the winning move was unique, that is to say, Alice *has* to make the stipulated move or else lose the game. He also remarked that in none of these cases did the strategy of taking the farthest asterisk lead to a win. Indeed he went further and proved that for *all* $3 \times n$ games, such a move loses. However, the attempt to discover a pattern in the different solutions failed. Gale went on to say:

> I expect the problem of finding explicit winning strategies may be hopeless, but I should think one might find a way of settling questions like the uniqueness of the first move.

He went on to consider briefly some generalizations: infinitely long rows or columns, and analogues in 3-dimensional and higher spaces. By the time his article actually appeared, the uniqueness question had been settled. Ken Thomson of the Bell Laboratories used a computer search and found several cases where uniqueness did not apply. The smallest such case was 8×10. Here both $(6,5)$ and $(9,4)$ secure the win for Alice. Thomson's work was subsequently confirmed by Michael Beeler of MIT's Artificial Intelligence Laboratory.

Another conjecture remains (as far as I can determine) open:

> The farthest asterisk strategy (n,m) always loses unless $m = 2$ or $n = 2$.

I am inclined to think this may well be true; see the remarks below.

At the conference where I first learned of Northeast, there was some discussion of cases where either m or n is infinite. (Although the number of asterisks is infinite in these cases, the number of *moves* is not; in all cases, the game terminates in finite time and so does not violate the terms of the winning strategy theorem.) The complete analysis of these cases had in fact earlier been supplied by Gale. Perhaps surprisingly, it is quite straightforward to dispose of all of them. There are four cases:

1. $1 \times \infty$ (or $\infty \times 1$)
2. $2 \times \infty$ (or $\infty \times 2$)
3. $m \times \infty$ (or $\infty \times m$), where $2 < m < \infty$
4. $\infty \times \infty$.

1. The case $m = 1$ is an instant win for Alice who simply plays $(2, 1)$.

2. The case $m = 2$ is, however, a surprise. It's a win for Bob! Gale left his readers to see why this is so, and I shall here follow his example. However, I shall make one further remark on this case. When we discussed the $2 \times n$ case, we found that Alice's winning strategy was to take the asterisk farthest from the poisoned one; however, when $n = \infty$, there is no such "farthest" one. It may be that this fact has some bearing on Gale's conjecture that this strategy only works when m or n equals 2.[10]

3. When m is finite and larger than 2, Alice wins by playing $(3, 1)$, and so landing Bob with the losing $m = 2$ position.

4. Finally when both m and n are infinite, Alice can win by playing $(3, 1)$ as in the previous case, or else by playing $(1, 3)$, and indeed she has a third way to the win: $(2, 2)$. So here we have a further case where the winning move is not unique.

In Cases 1 and 3, Alice loses if she does not play the right winning move, and in Case 4, she loses if she does not play one of the three available. I leave the details to the reader.

Now, although Gale was the first to publicize Northeast in this form, it turns out that in a sense he was preceded. The Dutch mathematician Fred Schuh, another keen game theorist, developed "The Game of Divisors", and explained it in a 1952 article in a Dutch journal. At first glance, it bears no relation to Northeast, but in fact it constitutes a generalization. I quote from Gardner's account.

> Two players agree on any positive integer, N. A list is made of all the divisors (including 1 and N), then players take turns crossing out a divisor and all *its* divisors. The person forced to take N loses. Planar Chomp corresponds to this game when N has exactly two prime divisors, solid Chomp to the game when N has three prime divisors, four-dimensional Chomp when N has four prime divisors, and so on.

He developed the connection with Northeast (Chomp) by means of an example. Suppose $N = 432 = 2^4 \times 3^3$. Raise each exponent by one to reach 5 and 4 and set up a 4×5 rectangle into which all the divisors of 432 are placed. This gives

16	8	4	2	1
48	24	12	6	3
144	72	36	18	9
432	216	108	54	27

[10] Alert readers will also observe that the proof that Alice has a winning strategy also breaks down when either m or n is infinite. In fact, not only does the proof fail but, as this case shows, the result is in fact false in these circumstances!

I leave it to the reader to check that all the divisors of 432 are accounted for and to discover the underlying pattern by which they are entered. Now I resume my quotation from Gardner's account.

> The equivalence of Chomp to the divisor game is now readily apparent. Moreover, any integer whose prime factors have the formula [$p^4 \times q^3$] will have the same Chomp field. Incredibly, most of the theorems discovered by David Gale for his game of Chomp (including the beautiful proof of first-player win) had been discovered by Schuh in Arithmetical form!

He concluded by offering a few remarks on the higher-dimensional versions of the game.

All in all, although the underlying theorem "concerns what could be thought of the simplest possible type of game we can imagine", this does not mean that it cannot lead to some intricate and difficult analyses!