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Wilkinson Polynomials

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Introduction

This article is about a family of polynomials introduced by James H. Wilkinson some five decades ago, which have the peculiar property that some of their zeros are extremely sensitive to small changes in the values of one or more of the coefficients.

Definition of the Wilkinson polynomials

If n is a non-negative integer, we define the Wilkinson polynomials $W(x, n)$ by

$$\begin{aligned} W(x,0) &= 1 \\ W(x,1) &= x-1 \\ W(x,2) &= (x-2)(x-1) \\ W(x,3) &= (x-3)(x-2)(x-1) \end{aligned}$$

and so on to give

$$W(x,n) = (x-n)W(x,n-1) \quad \text{for} \quad n > 0\,.$$

It is clear from this that $W(x, n)$ is a polynomial in x of degree n and that its zeros are at $x = 1, \ldots, n$. For example,

$$W(x,7) = x^7 - 28x^6 + 322x^5 - 1960x^4 + 6769x^3 - 13132x^2 + 13068x - 5040$$

has zeros at $x = 1, 2, 3, 4, 6, 5, 7$.

The special case $W(x, 20) = x^{20} - 210x^{19} + \cdots + 2432902008176640000$ is sometimes called **the** Wilkinson polynomial or Wilkinson's polynomial and denoted by $w(x)$. Perhaps of some historical interest is that the form originally used by Wilkinson was to write the terms as $(x + n)$, rather than as $(x - n)$ in his definition, but he later switched to the form used here.

Small changes to the Wilkinson polynomials

It had long been recognised that, if a polynomial had a multiple zero, a small change in one of the coefficients could result in a much larger change to these zeros. For example, consider the polynomial

$$P(x) = x^2 - 2x + 1 = (x-1)^2$$

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which has a double zero at $x = 1$. Let us now consider

$$Q(x) = P(x) - a^2$$

where a is a small number. The zeros of $Q(x)$ are clearly at $x = 1 + a$ and $x = 1 - a$. For example, if $a = 0.1$ a change in the constant coefficient of the polynomial from 1.0 to 0.99 has resulted in a change of one of the zeros from 1.0 to 1.1 and a change in the other zero from 1.0 to 0.9. Similar things can happen if two zeros are close together and a small change is made to one or more of the coefficients of the polynomial. What had not been appreciated fully until the advent of the Wilkinson polynomials is that small changes to the coefficients of a polynomial whose zeros are widely separated can also result in large changes to the zeros.

To illustrate this let us modify our Wilkinson polynomials by changing the coefficient of the second-highest power of x by a small amount. Thus we consider

$$\mathcal{M}(x, n, a) = W(x, n) + ax^{n-1}$$

where a is a constant. Clearly $\mathcal{M}(x, n, 0) = W(x, n)$.

Table 1 shows the zeros of $\mathcal{M}(x, 7, a)$ for various values of a.

$a = 0$	$a = 0.0001$	$a = 0.001$	$a = -0.001$
1	0.999999861	0.99999861	1.000001389
2	2.000053346	2.000534556	1.999467881
3	2.998487170	2.985372554	3.015825243
4	4.011578940	4.145004394	3.900810612
5	4.968043422	4.696557875	$5.475652121 + 0.2313509641i$
6	6.038523465	6.433991596	$5.475652121 - 0.2313509641i$
7	6.983213796	6.737540414	7.133590634

Table 1: The zeros of $\mathcal{M}(x, 7, a)$ for various values of a.

where $i = \sqrt{-1}$. In this table, only the first few decimal places have been shown. What is clear is that very small changes in the coefficient of x^6 have resulted in much larger changes to the zeros, particularly to the larger ones. Indeed, some of the zeros have become complex, involving $i = \sqrt{-1}$. There is nothing scary about this. For example, the polynomial $x^2 - 1$ has its zeros at $x = \pm 1$ whereas $x^2 + 1$ has its zeros at $x = \pm i$. If you have not encountered complex numbers yet, you will meet them later in your studies. They are used in many areas of science and engineering and are also discussed in a Wikipedia article referenced at the end of this article.

Why is this happening?

The original polynomial $W(x, 7)$ takes the value zero at $x = 1, 2, 3, 5, 6, 7$. Because it is a continuous function this means that $W(x, 7)$ is small in the neighbourhood of each of these points. When the additional term ax^6 is added the new polynomial $\mathcal{M}(x, 7, a)$ is not exactly zero at the zeros of the original polynomial. Thus the zeros are shifted. Now ax^6 grows very rapidly with x from a at $x = 1$ to $117649\,a$ at $x = 7$. Thus, for

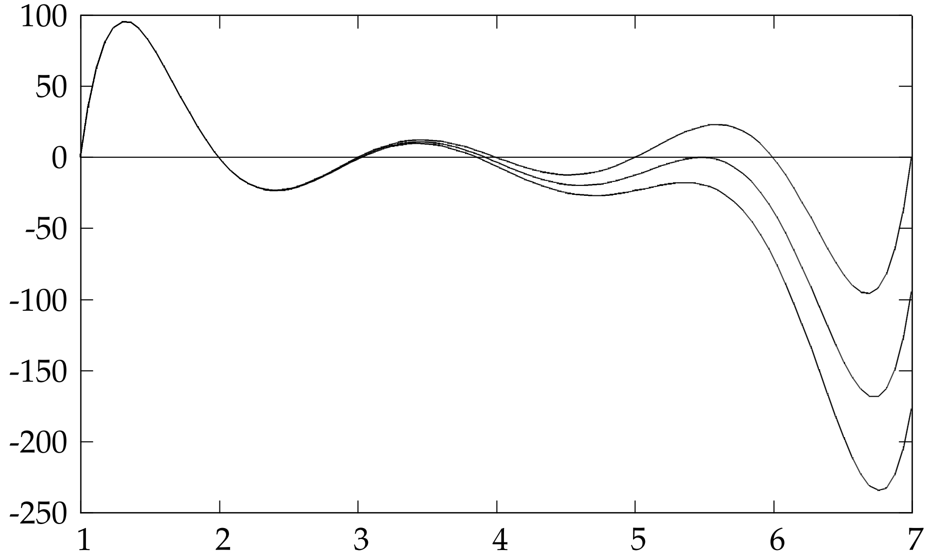

Figure 1: The upper curve is the Wilkinson polynomial $W(x, 7)$. The lower curve is $\mathcal{M}(x, 7, -0.0015)$. The middle curve is the critical case $\mathcal{M}(x, 7, -0.0008016332031)$ which has a double zero.

small values of a the zeros at $x = 1$ and $x = 2$ are altered only slightly whereas the higher ones experience greater changes. For example, if we start at $a = 0$ and then let a become more and more negative, we find that the zeros at 5 and 6 start to move towards one another. At the critical value of approximately $a = -0.0008016332031$ the two zeros coalesce to form a double zero at approximately $x = 5.479661255$. Beyond there, the curve no longer cuts the x-axis in this region and these two zeros become complex as we have seen. This behaviour is demonstrated in Figure 1.

Similar things happen for positive values of a. For example, zeros at 6 and 7 coalesce to a double zero at approximately $x = 6.593795626$ when $a = 0.001120388705$ approximately. Further increases in the magnitude of a lead to more zeros becoming complex.

Altering the coefficient of x^5 rather than x^6 leads to similar but less rapid changes because ax^5 grows more slowly than ax^6 as x increases. For Wilkinson polynomials of higher order, the changes are even more dramatic. The Wikipedia article mentioned below discusses $W(x, 20)$.

Why is this important?

A common circumstance in the practical application of mathematics to areas like science and engineering is to find the zeros of a function. You will all be familiar with the formula for finding the roots of a quadratic equation, i.e. the zeros of a polynomial of degree 2. There is a more complicated expression for the zeros of a polynomial of degree 3 and an horrendously complicated expression for the zeros of a polynomial of degree 4. It can be shown that there is no expression for the zeros of a general polynomial of degree 5 or greater. For these, numerical methods must be used as they must to find the solutions of an equation like $\cos x - x = 0$. You may be familiar with *Newton's method* (sometimes called the *Newton-Raphson method*) for finding a solution of the equation $f(x) = 0$. This starts from some initial estimate x_0 of the zero and constructs the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ and then finds where that tangent line cuts the x-axis. This is taken as the next estimate x_1 of the zero. The process is then repeated using x_1 instead of x_0 to give x_2, and so on. This therefore generates a sequence of numbers $x_0, x_1, x_2, \ldots x_n, \ldots$, which will sometimes approach the true but unknown zero. Indeed, there is a whole branch of mathematics known as *numerical analysis* devoted to devising and investigating methods which can be implemented on computers for finding approximate numerical solutions to a whole range of problems. One property of Newton's method in which we would be interested is to determine the circumstances under which it will converge to the zero we want. Loosely speaking, this will happen if x_0 is sufficiently close to the desired zero. For polynomials, there are specialised methods such as *Graeffe's method* and *Bairstow's method* which can be used.

We must bear in mind that computers and calculators are finite devices and cannot always represent real numbers exactly because they have mantissas of finite length. The inner workings of most computers use binary (base 2) or octal (base 8) or hexadecimal (base 16) arithmetic but the point can be made by using the more familiar decimal (base 10) representation of numbers. Suppose our computer has a mantissa of ten digits. A number such as $1/3$ would be represented in such a machine as 0.3333333333 which is not equal to $1/3$. Thus, any computer may not be able to hold the coefficients of a polynomial exactly and so we would be finding the zero of a polynomial slightly different from the one we intended. As the Wilkinson polynomial shows, some of the zeros of these two slightly different polynomials may be very different and so lead us to make false conclusions if, for example, we needed to know them for some engineering or scientific purpose.

Further Reading

As you might imagine, there is a large amount of material on the web about the matters discussed in this article. You might like to start at

http://en.wikipedia.org/wiki/Wilkinson's_polynomial

and follow some of the links there. For a short biography of James Wilkinson himself, you could start at

http://en.wikipedia.org/wiki/James_H._Wilkinson

For Graeffe's method, start at

http://en.wikipedia.org/wiki/Graeffe's_method

and for Bairstow's method try

http://en.wikipedia.org/wiki/Bairstow's_method

Newton's method is discussed in

http://en.wikipedia.org/wiki/Newton's_method

while complex numbers are discussed in

http://en.wikipedia.org/wiki/Complex_number

These web addresses are all correct at the time of writing but could possibly change over time.