Parabola Volume 48, Issue 3 (2012)

Problems 1401–1410

Q1401 Solve the recurrence relation

$$a(n) = 6a(n-1) - 9a(n-2)\ , \quad a(0) = 2\ , \quad a(1) = 21\ .$$

Comment. You can use the same method as in previous problems – see, for example, the solution to problem 1393 – but at one point you will find that things are a little different.

Q1402 Suppose that the three lines

$$y = ax + b \quad \text{and} \quad y = cx + d \quad \text{and} \quad y = ex + f$$

all have different gradients. Find conditions on a, b, c, d, e, f for the lines to intersect in a single point.

Q1403 Seven different real numbers are given. Prove that there are two of them, say x and y, for which

$$\frac{1 + xy}{x - y}$$

is greater than $\sqrt{3}$.

Q1404 In the game of poker, a pack of cards (consisting of the usual 52 cards) is shuffled and five cards are dealt to each player. A hand is referred to as "four of a kind" if it contains four cards of the same value and one other card. For example, ♠7, ♡7, ♢7, ♣7, ♢J constitutes four of a kind. Suppose that I deal two five–card hands from the same pack, one to myself and one to my opponent.

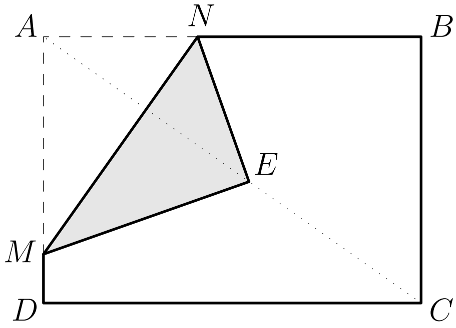
(a) What is the probability that my opponent has "four of a kind"?

(b) Suppose that I pick up my cards and discover that I have four of a kind. Is the probability that my opponent has four of a kind now greater or less than it was?

(c) Suppose on the other hand that I had picked up my cards and found that I *did not* have four of a kind. Is the probability that my opponent has four of a kind now greater or less than it was initially?

Q1405 A sheet of A4 paper has proportions $1 : \sqrt{2}$. Suppose such a sheet is grey on one side and white on the other, and label the corners A, B, C, D. The sheet is laid white side up and corner A is folded over, as shown in the diagram, in such a way that A, E, C are collinear and the length of the crease MN is equal to that of the side AD.

What proportion of the visible figure $MNBCD$ is grey?



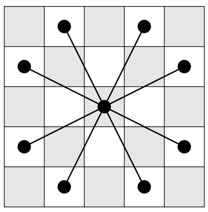
(And another question: **why** does a sheet of A4 paper have proportions $1:\sqrt{2}$?)

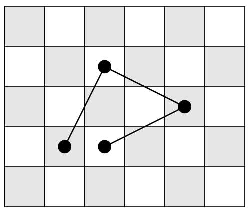
Q1406 Let n be a positive integer. Find polynomials $f(x)$ and $g(x)$ such that the only coefficients of these polynomials are 1 and -1, and

$$\frac{f(x)}{g(x)} = x^n - 2x^{n-1} + 1\ .$$

Q1407 Extend the game of Chomp (see Michael Deakin's article in this issue) to a three–dimensional array: whenever an asterisk is removed, so are all those which are to the north and/or east and/or higher than that one. The loser is the player who removes the asterisk in position $(1,1,1)$. If Alice, playing Bob on a $2\times2\times2$ grid, is the one to move first, find all her winning moves (if any).

Q1408 In the game of chess, a knight moves two squares horizontally or vertically and then one square in a perpendicular direction. So, as long as the edges of the board do not get in the way, the knight has at any stage eight possible moves, as shown in the first diagram. From the second diagram we can see that it is possible for a knight (making more than one move) to move from one square to that immediately to the right.





Now suppose that a "superknight" moves in a similar fashion but with three steps in one direction and eight in a perpendicular direction. Also suppose that the board is infinitely large, so that there are no edges to get in the way. Can a superknight move from one square to that immediately to the right?

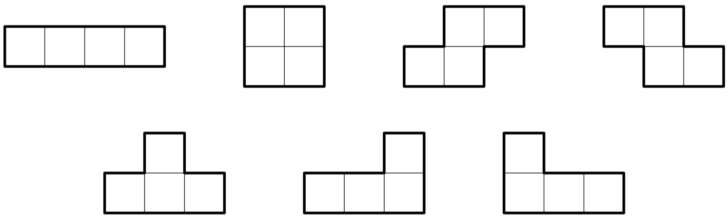
Q1409 Suppose that the following statement (related to the ABC Conjecture – see the editorial in this issue) is true:

if a, b, c are positive integers such that a, b have no common factor and $a + b = c$; and if P is the product of all prime factors, taken once each, of a, b, c; then $c < P^2$.

Prove the following result, which is (almost) Fermat's Last Theorem:

there are no positive integers x, y, z and n, with $n > 5$, such that $x^n + y^n = z^n$.

Q1410 A **tetromino** is a shape constructed by joining four squares of unit size along their edges. There are seven such shapes (which will be familiar to anyone who has ever played Tetris) and they are shown below.



The shapes obviously have a total area of 28. Can they be arranged so as to completely cover the following shape of area 28?