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Solutions 1391–1400

Q1391 Jack looked at the clock next to his front door as he left home one afternoon to visit Jill and watch a TV programme. Arriving exactly as the programme started, he set out for home again when it finished one hour later. As he did so he looked at her clock and noticed that it showed the same time as his had done when he left home. Puzzling over how Jill's clock could be so wrong, Jack travelled home at half the speed of his earlier journey. When he arrived home he saw from his clock that the whole expedition had taken two hours and fifteen minutes. He still hadn't worked out about Jill's clock and so he called her up on the phone. Jill explained that her clock was actually correct (as was Jack's), but it was an "anticlockwise clock" on which the hands travel in the opposite direction from usual. Jack had been in such a hurry to leave that he hadn't noticed the numbers on the clock face going the "wrong" way around the dial. At what time did Jack leave home?

SOLUTION Jack's travel time home was twice that of his outward journey; the total travel time, plus the 60 minutes' visit, adds up to 2 hours 15 minutes. So Jack's outward travel time was 25 minutes, and the time when he looked at Jill's clock was 1 hour 25 minutes after looking at his own. Since it then appeared to show the time at which Jack left, double this time plus the 1 hour 25 minute stay must add up to 12 hours (if you have trouble seeing why, draw pictures of the two clocks). So the time at which Jack left was $5:17:30$.

Comment. The times involved could have added up to 24 hours instead of 12, in which case Jack's departure time would have been $11:17:30$. However this does not fit in with the information that he "left home one afternoon".

Q1392 Find all real numbers x which satisfy the equation

$$\lfloor x \rfloor - \{2x\} + \lceil 3x \rceil = 5 ,$$

where we write $\lfloor x \rfloor$ for x rounded to the integer below, $\lceil x \rceil$ for x rounded to the integer above, and $\{x\}$ for x rounded to the nearest integer, with halves rounding upwards.

SOLUTION Following the solution to problem 1384, let $a = \lfloor x \rfloor$ and $b = \{2x\}$ and $c = \lceil 3x \rceil$. Then a, b, c are integers and we can write

$$x = a + \alpha , \quad 2x = b + \beta , \quad 3x = c - \gamma \tag{1}$$

where

$$0 \le \alpha < 1 , \quad -\tfrac{1}{2} \le \beta < \tfrac{1}{2} , \quad 0 \le \gamma < 1 . \tag{2}$$

We have immediately

$$a - b + c = 5 ,$$

and from (1) we obtain

$$2a + 2\alpha = b + \beta \quad \text{and} \quad 3a + 3\alpha = c - \gamma \; . \tag{3}$$

Solving the last three equations to find a, b, c in terms of α, β, γ is easy (exercise!) and gives the results

$$a = \tfrac{1}{2}(5 - \alpha - \beta - \gamma)\,, \;\; b = 5 + \alpha - 2\beta - \gamma\,, \;\; c = \tfrac{1}{2}(15 + 3\alpha - 3\beta - \gamma)\,.$$

But now the inequalities (2) imply

$$1\tfrac{1}{4} < a \le 2\tfrac{3}{4}\,, \;\; 3 < b < 7\,, \;\; 6\tfrac{1}{4} < c < 9\tfrac{3}{4}\,,$$

and since a, b, c are integers we have

$$a = 2\;, \quad b = 4 \text{ or } 5 \text{ or } 6\;, \quad c = 7 \text{ or } 8 \text{ or } 9$$

with $c = b + 3$. Returning to equations (3) and solving for α we obtain

$$\alpha = \tfrac{1}{2}(b - 2a + \beta) \quad \text{and} \quad \alpha = \tfrac{1}{3}(c - 3a - \gamma)\;,$$

and using one last time the inequalities (2) together with the known possibilities for a, b, c gives restrictions on the value of α. Not forgetting that we already know $0 \le \alpha < 1$, there are three possibilities:

- $a = 2$, $b = 4$, $c = 7$, so $-\frac{1}{4} \le \alpha < \frac{1}{4}$ and $0 < \alpha \le \frac{1}{3}$, so
$$0 < \alpha < \tfrac{1}{4}\;;$$
- $a = 2$, $b = 5$, $c = 8$, so $\frac{1}{4} \le \alpha < \frac{3}{4}$ and $\frac{1}{3} < \alpha \le \frac{2}{3}$, so
$$\tfrac{1}{3} < \alpha \le \tfrac{2}{3}\;;$$
- $a = 2$, $b = 6$, $c = 9$, so $\frac{3}{4} \le \alpha < \frac{5}{4}$ and $\frac{2}{3} < \alpha \le 1$, so
$$\tfrac{3}{4} \le \alpha < 1\;.$$

Since $x = 2 + \alpha$, the solutions of the equation must satisfy

$$2 < x < 2\tfrac{1}{4} \quad \text{or} \quad 2\tfrac{1}{3} < x \le 2\tfrac{2}{3} \quad \text{or} \quad 2\tfrac{3}{4} \le x < 3\;.$$

There are other conditions which could potentially restrict the solutions still further, so it is necessary to check that all these values really work; however this is not difficult, and all of these x values are in fact solutions.

Q1393 The sequence of numbers $a(1), a(2), \ldots$ in the solution to problem 1385 (previous issue) has the properties $a(1) = 5$, $a(2) = 26$ and

$$a(n) = 6a(n-1) - 4a(n-2) \quad \text{for} \quad n \geq 3 \,.$$

Use the method of the article in the previous issue to find a formula for $a(n)$ directly in terms of n, and hence check the answer to problem 1385.

SOLUTION As in the article, we wish to rewrite the recurrence in the form

$$[a(n) - ra(n-1)] = s[a(n-1) - ra(n-2)] \,, \tag{1}$$

that is,

$$a(n) = (r+s)a(n-1) - rsa(n-2) \,.$$

Therefore we need $r + s = 6$, $rs = 4$; r and s are roots of the equation $x^2 - 6x + 4 = 0$, and we may take

$$r = 3 + \sqrt{5} \,, \quad s = 3 - \sqrt{5} \,.$$

Equation (1) now gives

$$\begin{aligned} a(n) - ra(n-1) &= s\,[a(n-1) - ra(n-2)] \\ &= s^2\,[a(n-2) - ra(n-3)] \\ &= \cdots \\ &= s^{n-2}\,[a(2) - ra(1)] \end{aligned}$$

and hence

$$\begin{aligned} a(n) - ra(n-1) &= s^{n-2}\,[a(2) - ra(1)] \\ ra(n-1) - r^2a(n-2) &= rs^{n-3}\,[a(2) - ra(1)] \\ \cdots &= \cdots \\ r^{n-2}a(2) - r^{n-1}a(1) &= r^{n-2}\,[a(2) - ra(1)] \,. \end{aligned}$$

Adding all of these equations, most of the left-hand side disappears, and we add up a geometric series on the right-hand side (being careful with the number of terms):

$$\begin{aligned} a(n) - r^{n-1}a(1) &= [a(2) - ra(1)]\,s^{n-2}\,\frac{(r/s)^{n-1} - 1}{(r/s) - 1} \\ &= [a(2) - ra(1)]\,\frac{r^{n-1} - s^{n-1}}{r - s} \,. \end{aligned}$$

Solving for $a(n)$ and collecting terms, then substituting the known values of $r, s, a(1)$ and $a(2)$, yields

$$\begin{aligned} a(n) &= \frac{a(2) - sa(1)}{r - s}\,r^{n-1} - \frac{a(2) - ra(1)}{r - s}\,s^{n-1} \\ &= \frac{11 + 5\sqrt{5}}{2\sqrt{5}}\,r^{n-1} - \frac{11 - 5\sqrt{5}}{2\sqrt{5}}\,s^{n-1} \\ &= \left(\frac{1}{2} + \frac{1}{\sqrt{5}}\right)(3 + \sqrt{5})^n + \left(\frac{1}{2} - \frac{1}{\sqrt{5}}\right)(3 - \sqrt{5})^n \,. \end{aligned}$$

Substituting $n = 10$ and doing lots of algebra (or using a computer algebra system) gives $a(10) = 14\,672\,384$, confirming the result of solution 1385.

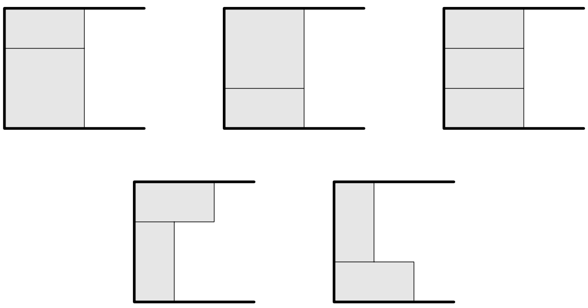
NOW TRY problem 1401.

Q1394 Consider the problem of covering a hallway of width 3 metres with carpet tiles, if the tiles available are 2 metre by 2 metre squares and 2 metre by 1 metre rectangles. (See the article in the previous issue for a similar problem.)

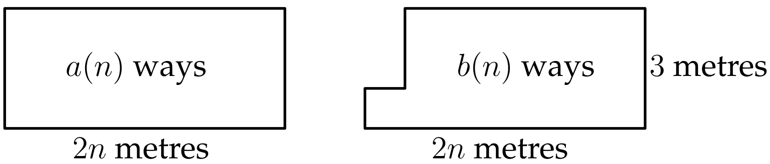
(a) How many ways are there of doing this if the length (in metres) of the hallway is odd?

(b) Let $a(n)$ be the number of ways of carpeting a 3 metre by $2n$ metre hallway. Find a recurrence relation for $a(n)$ and solve it to obtain a direct formula for $a(n)$.

SOLUTION For the first part we note that the given tiles, in any combination, will always cover an even area. Therefore there is no way to carpet the hallway if it has odd length. For a hallway of length $2n$ metres there are five ways, as shown in the diagrams, to cover the end of the hallway.



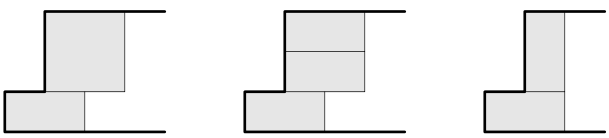
Now let $a(n)$ and $b(n)$ be the number of ways to carpet regions with the shapes and dimensions shown.



The first set of diagrams shows that

$$a(n) = 3a(n-1) + 2b(n) \quad \text{for } n \geq 2, \tag{1}$$

while the following



give

$$b(n) = 2b(n-1) + a(n-1) \quad \text{for } n \geq 2. \tag{2}$$

Here we have not a simple recurrence relation, but a system of two simultaneous recurrences in two unknowns. To solve it, replace n by $n-1$ in (1),

$$a(n-1) = 3a(n-2) + 2b(n-1) \quad \text{for } n \geq 3, \tag{3}$$

and then subtract twice equation (3) from equation (1), giving

$$a(n) - 2a(n-1) = 3a(n-1) - 6a(n-2) + 2b(n) - 4b(n-1)$$

for $n \geq 3$. We can now use (2) to eliminate the b terms from this; after tidying up a bit we get

$$a(n) = 7a(n-1) - 6a(n-2)\ .$$

Using the first set of diagrams, we have the initial condition $a(1) = 5$; similar pictures (exercise!) show that $b(1) = 1$; then using (2) gives $b(2) = 7$ and $a(2) = 29$. So we have to solve the initial value problem

$$a(n) = 7a(n-1) - 6a(n-2)\ , \quad a(1) = 5\ , \quad a(2) = 29\ ,$$

and applying the method of the previous problem (do it!) yields the answer

$$a(n) = \frac{4 \times 6^n + 1}{5}\ .$$

Q1395 Find all real values of a and b such that the parabolas

$$y = x^2 + ax + b \quad \text{and} \quad y = -x^2 + bx + a$$

do not intersect.

SOLUTION The curves do not intersect whenever the equation

$$x^2 + ax + b = -x^2 + bx + a$$

has no solution. This equation is equivalent to the quadratic

$$2x^2 + (a-b)x - (a-b) = 0\ ,$$

and it has no (real) solution when the discriminant is negative,

$$(a-b)^2 + 8(a-b) < 0\ ,$$

which can be rewritten as

$$(a-b)(a-b+8) < 0\ .$$

If the product is negative then one factor must be positive and the other negative. But $a-b+8$ is clearly the bigger of the two factors, so it is positive and the other is negative. Thus

$$a - b < 0 \quad \text{and} \quad a - b + 8 > 0\ ,$$

that is,

$$0 < b - a < 8 \ ,$$

and this is the condition for the parabolas not to intersect.

NOW TRY problem 1402.

Q1396 Consider a triangle with sides of length a, b, c, and let l be the length of the line which bisects the angle opposite the side of length a. Prove that

$$l^2 = \frac{4bc}{(b+c)^2}\, s(s-a) \ ,$$

where s is the semi perimeter of the triangle. Deduce that the area of the triangle is at least

$$\frac{l_a l_b l_c}{8s} \ ,$$

where l_a, l_b, l_c are the lengths of the three angle–bisectors.

SOLUTION Let the angle opposite the side of length a be 2α. The area of the triangle is the sum of the two smaller triangular areas formed by the angle–bisector:

$$\frac{bc \sin 2\alpha}{2} = \frac{bl \sin\alpha}{2} + \frac{cl \sin\alpha}{2} \ .$$

Now using the double–angle formula $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and cancelling $\sin\alpha$ (which is not zero since $0 < \alpha < \pi$), we can simplify this equality to get

$$2bc\cos\alpha = (b+c)l \ . \qquad (*)$$

On the other hand, we can use the cosine rule and another double–angle formula to obtain

$$a^2 = b^2 + c^2 - 2bc\cos 2\alpha = b^2 + c^2 - 2bc(2\cos^2\alpha - 1) \ ;$$

multiplying both sides by bc and substituting from equation $(*)$ yields

$$\begin{aligned} bca^2 &= bc(b^2 + c^2 + 2bc) - (2bc\cos\alpha)^2 \\ &= bc(b+c)^2 - (b+c)^2 l^2 \ . \end{aligned}$$

Therefore

$$\begin{aligned} l^2 &= \frac{bc((b+c)^2 - a^2)}{(b+c)^2} \\ &= \frac{bc(b+c+a)(b+c-a)}{(b+c)^2} \\ &= \frac{bc(2s)(2s-2a)}{(b+c)^2} \\ &= \frac{4bc}{(b+c)^2}\, s(s-a) \end{aligned}$$

as required. Finding a similar formula for the bisectors of the other two angles, and multiplying all three results, gives

$$\begin{aligned} l_a^2 l_b^2 l_c^2 &= \frac{4bc}{(b+c)^2}\, s(s-a)\, \frac{4ca}{(c+a)^2}\, s(s-b)\, \frac{4ab}{(a+b)^2}\, s(s-c) \\ &= \frac{64a^2b^2c^2}{(b+c)^2(c+a)^2(a+b)^2}\, s^2[s(s-a)(s-b)(s-c)]\ . \end{aligned}$$

But by Heron's formula, the expression in square brackets is just the square of the area of the triangle:

$$l_a^2 l_b^2 l_c^2 = \frac{64a^2b^2c^2}{(b+c)^2(c+a)^2(a+b)^2}\, s^2 A^2\ ,$$

and so

$$A = \frac{b+c}{a}\, \frac{c+a}{b}\, \frac{a+b}{c}\, \frac{l_a l_b l_c}{8s}\ .$$

Finally, since a, b, c are the sides of a triangle, each of the first three fractions in this expression is at least 1; hence

$$A \geq \frac{l_a l_b l_c}{8s}$$

as claimed.

Q1397 Find a number consisting of four different digits in ascending order, such that if the digits are written in reverse to form another four-digit number, and if the two four–digit numbers are added, the result is 9218.

SOLUTION If the digits, in ascending order, are a, b, c, d, then we have

$$(1000a + 100b + 10c + d) + (1000d + 100c + 10b + a) = 9218\ , \qquad (1)$$

that is,

$$1001(a + d) + 110(b + c) = 9218\ .$$

Looking at the units digits in (1) we have $a + d = 8$, or $a + d = 18$ with a "carry" of 1 into the tens column; but the latter is impossible since two different digits cannot add up to 18. So $a + d = 8$, and equation (2) gives $b + c = 11$. Since a, b, c, d are single digits in increasing order, and $a \neq 0$ (otherwise we would not have two four–digit numbers), the only possibility is $a = 1$, $d = 7$, $b = 5$, $c = 6$. So the required number is 1567.

Q1398 Let p be a polynomial and suppose that $p(x)$ is a factor of $p(x^2)$. Prove that the equation $p(x) = 0$ has no real solutions except possibly 0, 1 or -1.

SOLUTION If $p(x)$ is a factor of $p(x^2)$ then there is a polynomial $q(x)$ such that

$$p(x^2) = p(x)q(x)\ .$$

Now let a be a solution of $p(x) = 0$; that is, $p(a) = 0$. Then

$$p(a^2) = p(a)q(a) = 0\, q(a) = 0\ ,$$

so a^2 is a solution of the same equation. Repeating the procedure, we find that $a, a^2, a^4, a^8, \ldots$ are all solutions of $p(x) = 0$. But a polynomial equation can only have a finite number of solutions, so (at least) two of these solutions must be the same:

$$a^k = a^l \,, \quad k \neq l \,.$$

Thus $a^k(1 - a^{l-k}) = 0$ and we have either $a^k = 0$ or $a^{l-k} = 1$. So a can only be 0, 1 or -1.

Q1399 There are n double seats in a railway carriage, all occupied. People get up and leave, one at a time, in random order. Find the probability that after the last pair is broken there remain exactly k people in the carriage.

SOLUTION Imagine that we make a video of the whole procedure, and then view the video backwards, so that instead of getting up and leaving, people arrive and sit down. We can then see that the answer to the problem posed is the same as to the following: if people arrive one by one in the empty carriage and choose seats at random, find the probability that the first k people form no pairs, and person $k + 1$ sits next to one of the people already there. The probability is clearly zero if $k > n$, so we suppose that $k \leq n$. Now the first person clearly does not form a pair; the second avoids a pair with probability $(2n-2)/(2n-1)$; the third avoids a pair with probability $(2n-4)/(2n-2)$; and so on. The probability that the first k people form no pairs is

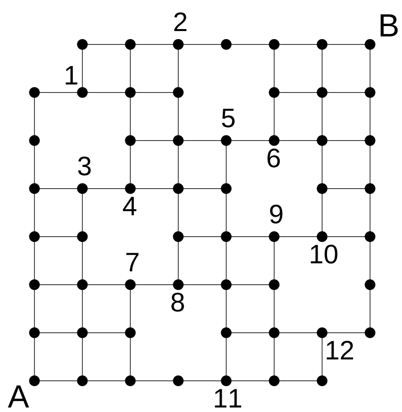
$$\frac{2n-2}{2n-1}\,\frac{2n-4}{2n-2}\cdots\frac{2n-2(k-1)}{2n-(k-1)} \,.$$

There are now $2n - k$ unoccupied seats, including k which are next to an occupied seat; so the probability that person $k + 1$ *does* form a pair is $k/(2n-k)$. The final answer is the above product times this fraction; it can be simplified in various ways to get

$$\frac{2n}{2n}\,\frac{2n-2}{2n-1}\,\frac{2n-4}{2n-2}\cdots\frac{2n-2(k-1)}{2n-(k-1)}\,\frac{k}{2n-k}$$

$$= k\,2^k\,\frac{n!}{(n-k)!}\,\frac{(2n-k-1)!}{(2n-k)!} \,.$$

Q1400 In the following diagram, in how many ways can one travel along the lines from point A to point B if one may only move upwards and to the right?



SOLUTION To help explain the solution, we have labelled some more points in the diagram. It is not hard to see that any path from A to B must fall into one of the following categories:

A–1–2–B	A–3–4–2–B	A–3–4–5–6–B
A–7–8–2–B	A–7–8–5–6–B	A–7–8–9–10–B
A–11–5–6–B	A–11–9–10–B	A–11–12–B

Consider, for example, the number of ways to go from A to 7. We have to take four steps of which two are upwards and two are to the right; these four steps may be taken in any order and so the number of possibilities is the binomial coefficient $\binom{4}{2}$, sometimes written $^4C_2$ or $C(4,2)$. Applying the same ideas to all the other "rectangles" in the diagram and evaluating the binomial coefficients by means of Pascal's triangle, the number of possible routes in each of the above nine categories is

$\binom{3}{1} = 3$	$\binom{5}{1}\binom{4}{1} = 20$	$\binom{5}{1}\binom{3}{1}\binom{4}{2} = 90$
$\binom{4}{2} = 6$	$\binom{4}{2}\binom{4}{1}\binom{4}{2} = 144$	$\binom{4}{2}\binom{3}{1}\binom{5}{1} = 90$
$\binom{4}{2} = 6$	$\binom{4}{1}\binom{5}{1} = 20$	$\binom{3}{1} = 3$

and the total is 382.