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The Generalization of the Disk Method

Jihoon Kim [1]

Abstract

This method generalizes the disk method of integration to find the volume of a rotated function by utilizing a differential dependent on both the axis of rotation and the rotated function.

1 Introduction

The common textbook definition explains the Disk Method as a three-dimensional extension of the Riemann Sums on the x-axis or y-axis. However, rather than creating an infinite sum of rectangles with infinitesimal width, the Disk Method creates an infinite sum of disks with infinitesimal height, with the central axis on either the x-axis or the y-axis. It is obvious that this is a special case. Here, a generalization of the Disk Method that works for any axis is provided.

2 Derivation

To find the volume of a function rotated about an axis, a general formula can be applied:

$$\pi \int_a^b [R(x)]^2 dh,$$

where a and b are the limits of rotation, $R(x)$ is the radius between the function and the axis of rotation, and dh is the height differential dependent on the axis of rotation. The common textbook formula replaces the dh with dx, as the height differential for the standard axis simplifies to dx. Given an axis of rotation defined by the line $ax + by = c$, the radius, dependent on the function to be rotated and the axis of rotation, is defined by the formula for the distance from a point to a line:

$$R(x) = \frac{|a \cdot x + b \cdot f(x) + c|}{\sqrt{a^2 + b^2}}.$$

The height $h(x)$ can be found by using the Pythagorean Theorem twice. First, the

[1] Jihoon Kim is a research intern in the Feinberg School of Medicine, Northwestern University, Chicago

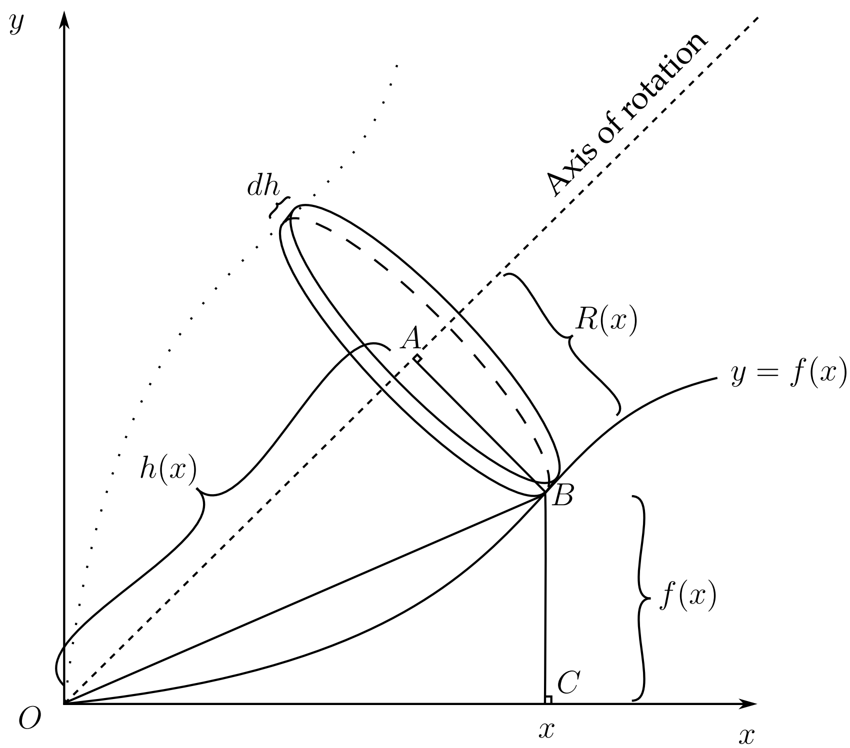

Figure 1: Derivation of dh

hypotenuse $\overline{\text{CB}}$ of right triangle $\triangle$CBD can be expressed as $\sqrt{x^2 + f(x)^2}$. Segment $\overline{\text{CB}}$ is also the hypotenuse of $\triangle$ABC. Thus, given $\overline{\text{CB}}$ and leg $\overline{\text{AB}}$, defined by $R(x)$, $h(x)$ can be expressed as $\sqrt{x^2 + f(x)^2 - [R(x)]^2}$. Then $h(x)$ can be differentiated so that dh can be expressed as

$$\frac{d}{dx}\sqrt{x^2 + f(x)^2 - [R(x)]^2}.$$

Replacing the constants with respective equivalent functions will solve for volumes of rotations about any linear axis of rotation.

3 Example

To find the volume of the solid resulting from the function $y = \frac{1}{2}x$ about the axis of rotation $y = x$ from $x = 0$ to $x = 6$, we could use the general formula. In this case, the constants a and b are respectively 1 and -1, and c is 0. Thus

$$R(x) = \frac{|1 \cdot x - \frac{1}{2} \cdot x + 0|}{\sqrt{1^2 + (-1)^2}} = \frac{\frac{1}{2} \cdot x}{\sqrt{2}} = \frac{x}{2\sqrt{2}}.$$

Next, the dh can be expressed as

$$\frac{d}{dx}\sqrt{x^2 + \left[\frac{1}{2} \cdot x\right]^2 - \left[\frac{x}{2\sqrt{2}}\right]^2} = \frac{d}{dx}\sqrt{x^2 + \frac{x^2}{4} - \frac{x^2}{8}} = \frac{d}{dx}\sqrt{\frac{9x^2}{8}} = \frac{3}{2\sqrt{2}}dx.$$

Replacing respective constants and functions with equivalents in the general formula gives the volume:

$$\pi \int_0^6 \Big[\frac{x}{2\sqrt{2}}\Big]^2 \frac{3}{2\sqrt{2}} dx \ = \ \pi \int_0^6 \frac{3x^2}{16\sqrt{2}} dx = \frac{27\pi}{2\sqrt{2}}.$$

This can be verified using geometry. By rotating the function $y = \frac{1}{2}x$ about the axis of rotation $y = x$, a cone is created. The cone has the center of its base at point $\{\frac{9}{2}, \frac{9}{2}\}$, radius, given by the Euclidean distance formula, of $\frac{3}{\sqrt{2}}$ and a height of $\frac{9}{\sqrt{2}}$. Then given the geometrical formula for the volume of a cone $V_{cone} = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$, the volume of the generated solid is

$$\frac{1}{3} \cdot \pi \cdot (\frac{3}{\sqrt{2}})^2 \cdot \frac{9}{\sqrt{2}} = \frac{27\pi}{2\sqrt{2}},$$

which was given by the disk method generalization.

4 Conclusion

To summarize, the disk method of rotation can be generalized to compute the volumes of functions rotated about non-standard axes. By creating radius functions and height differentials dependent on the rotated function and the axis of rotation, the disk method can be generalized to accomodate such cases. Further, this method opens up avenues for generalizing Riemann Sum and Lebesgue Integration about non-standard axes.