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Problems 1411–1420

Parabola would like to thank Sin Keong Tong for contributing problem 1413.

Q1411 How many three–digit numbers are there which do not contain any digit more than once? What do you get if you add them all up?

Q1412 Find all solutions of the equation

$$x^4 + 4x^3 - 6x^2 - 20x + 13 = 0 .$$

Q1413 By arranging the digits of the number 2013 we obtain twenty–four different numbers 0123, 0132, $\ldots$, 2013, $\ldots$, 3210, where the first digit of a number **is** permitted to be zero. Find the smallest possible value of

$$|x - 0123| + |x - 0132| + \cdots + |x - 2013| + \cdots + |x - 3210| ,$$

where x is a real number.

Q1414 Let a and b be positive integers, and consider a "knight–like" piece which moves on a chessboard a squares up, down, left or right and then b squares in a perpendicular direction. We shall refer to such a piece as an (a, b)–*superknight*: for example, the piece described in problem 1408 is a $(3, 8)$–superknight, while a $(2, 1)$–superknight is just an ordinary knight.

Find all values of a and b for which an (a, b)–superknight can move from one square to that immediately to the right. As in problem 1408, we assume that the board is infinitely large, so that there are no edges to get in the way.

Q1415 How often is the sun directly overhead at the equator? Once a day? Twice a day? Something else? Explain your answer!

Q1416 A chord cuts off an arc of a circle. If the chord length is c, the arc length s and the maximum (perpendicular) distance from the chord to the arc h, prove that

$$\frac{2h}{c} = \tan \frac{2hs}{c^2 + 4h^2} .$$

Q1417 If n is a positive integer, let

$$P = 1^1 \times 2^2 \times 3^3 \times \cdots \times n^n .$$

Without using a calculator, prove that if $n = 17$ then P is larger than a googol. (Remember that a googol is the number consisting of a 1 followed by a hundred 0s, that is, 10^{100}.) For a harder question, prove that in fact if $n = 15$ then P is already larger than a googol – still without using a calculator of course!

Q1418 Consider a square lattice of points consisting of m rows and n columns. In how many different ways is it possible to choose m points, one from each row, which are collinear? The case $m = 3$, $n = 10$ is illustrated, together with one possible set of three points.

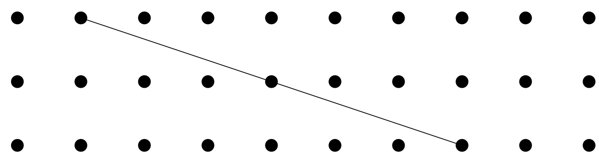

Figure 1: Example case where $m = 3$ and $n = 10$, showing one possible choice in which 3 (i.e. m) points are collinear.

Q1419 Given an integer $n \geq 2$, what is the greatest number that can be obtained by writing n as a sum of positive integers and multiplying those integers? For example, if $n = 2013$ we could write $n = 1006 + 1007$ and obtain the product $1006 \times 1007 = 1013042$, or $n = 1000 + 1000 + 13$ giving $1000 \times 1000 \times 13 = 13000000$, but neither of these is the greatest possible product.

Q1420 Let n be a positive integer. Show that if n is odd, then it is **not possible** to find polynomials $f(x)$ and $g(x)$ such that the only coefficients of these polynomials are 1 and -1, and

$$\frac{f(x)}{g(x)} = x^n - x^{n-1} + 1 .$$

Note that in particular, $f(x)$ and $g(x)$ may not have any zero coefficients: for example, $g(x) = x^3 - x + 1$ does not meet the requirements of the question.