Parabola Volume 49, Issue 1 (2013)

Solutions 1400–1410

Q1400 In the last issue, the end of the solution to problem 1400 was inadvertently omitted: adding up the numbers found at the end of the printed solution (page 59), we see that the total number of ways to get from A to B is 382.

Q1401 Solve the recurrence relation

$$a(n) = 6a(n-1) - 9a(n-2)\ , \quad a(0) = 2\ , \quad a(1) = 21\ .$$

SOLUTION Using the method of previous problems (for example, problem 1393), we rewrite the recurrence as

$$a(n) - ra(n-1) = s[a(n-1) - ra(n-2)]$$

with $r = s = 3$.

Comment. Notice that in this example, unlike those we have seen before, r and s are the same. It turns out that this will make a bit of difference to the solution. Continuing, we have

$$\begin{aligned} a(n) - 3a(n-1) &= 3\,[a(n-1) - 3a(n-2)] \\ &= 3^2\,[a(n-2) - 3a(n-3)] \\ &= \cdots \\ &= 3^{n-1}\,[a(1) - 3a(0)] \\ &= 5 \times 3^n \end{aligned}$$

and therefore

$$\begin{aligned} a(n) - 3a(n-1) &= 5 \times 3^n \\ 3a(n-1) - 3^2a(n-2) &= 5 \times 3^n \\ \cdots &= \cdots \\ 3^{n-1}a(1) - 3^na(0) &= 5 \times 3^n \end{aligned}$$

(notice that this time all the right-hand sides are the same). Adding all these equations and observing that most of the terms on the left-hand side cancel, we have

$$a(n) - 2 \times 3^n = n(5 \times 3^n)\ ,$$

which gives the final solution

$$a(n) = (5n+2)3^n\ .$$

Q1402 Suppose that the three lines

$$y = ax + b \quad \text{and} \quad y = cx + d \quad \text{and} \quad y = ex + f$$

all have different gradients. Find conditions on a, b, c, d, e, f for the lines to intersect in a single point.

SOLUTION The lines will intersect in a single point provided that the x–coordinate of the intersection of the first and second is the same as the x–coordinate of the intersection of the second and third, that is,

$$\frac{d-b}{a-c} = \frac{f-d}{c-e} \ .$$

Note that since the gradients are all different, the denominators of the fractions are not zero. The equation can be rewritten in a form which is perhaps a little simpler,

$$ad + be + cf = af + bc + de \ .$$

Q1403 Seven different real numbers are given. Prove that there are two of them, say x and y, for which

$$\frac{1+xy}{x-y}$$

is greater than $\sqrt{3}$.

SOLUTION The arctangents of the seven numbers are seven different real numbers lying strictly between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. That is, we have now seven numbers in an interval of length π, and so there must be two of them which are separated by less than $\frac{1}{6}\pi$. If $\arctan x$ is the larger of these two numbers and $\arctan y$ is the smaller, we have

$$0 < \arctan x - \arctan y < \frac{\pi}{6} \ .$$

Now $\tan$ is an increasing function between 0 and $\frac{1}{6}\pi$ and so

$$0 < \tan\bigl(\arctan x - \arctan y\bigr) < \frac{1}{\sqrt{3}} \ ;$$

using the "tan of a difference" formula and the fact that $\tan(\arctan x) = x$ gives

$$0 < \frac{x-y}{1+xy} < \frac{1}{\sqrt{3}} \ ;$$

finally, since the first inequality shows that we are dealing with positive numbers, we may take reciprocals in the second inequality to give

$$\frac{1+xy}{x-y} > \sqrt{3}$$

as claimed.

Correct solution received from an anonymous correspondent.

Q1404 In the game of poker, a pack of cards (consisting of the usual 52 cards) is shuffled and five cards are dealt to each player. A hand is referred to as "four of a kind" if it contains four cards of the same value and one other card. For example, $\spadesuit 7$, $\heartsuit 7$, $\diamondsuit 7$, $\clubsuit 7$, $\diamondsuit$J constitutes four of a kind. Suppose that I deal two five–card hands from the same pack, one to myself and one to my opponent.

(a) What is the probability that my opponent has "four of a kind"?

(b) Suppose that I pick up my cards and discover that I have four of a kind. Is the probability that my opponent has four of a kind now greater or less than it was?

(c) Suppose on the other hand that I had picked up my cards and found that I *did not* have four of a kind. Is the probability that my opponent has four of a kind now greater or less than it was initially?

SOLUTION

(a) The number of five–card hands which may be dealt from a 52–card pack is the binomial coefficient $C(52, 5)$, sometimes written $\binom{52}{5}$ or ${}^{52}C_5$. To choose a "four of a kind" hand we must choose the value for the "four" (13 possibilities), then choose one further card (48 possibilities). So the probability that my opponent has "four of a kind" is

$$\frac{13 \times 48}{C(52, 5)} = 0{\cdot}0002400960384 \, .$$

(b) Before doing the calculations, it's interesting to try to work out how the answer should compare with part (a). It is fairly plausible that if we increase the number of cards in each suit (for example, consider a 100–card pack with 25 different values in each suit) the likelihood of "four of a kind" should decrease. In the present scenario, an entire four (and one extra card) is ruled out from my opponent's hand; so, more or less, his hand has come from a smaller pack (of 48 cards); so we should expect his probability of "fours" to be greater than in the previous case.

To do this more precisely, since we know that I have "fours", my opponent's hand must be chosen from a 47–card pack consisting of 11 groups of 4 and one group of 3. The number of possible hands is $C(47, 5)$; to choose "fours" we pick a four (11 possibilities) and then one other card (43 possibilities). So the probability that my opponent has "fours" in this case is

$$\frac{11 \times 43}{C(47, 5)} = 0{\cdot}0003083564601 \, ;$$

as expected, this is bigger than the previous answer.

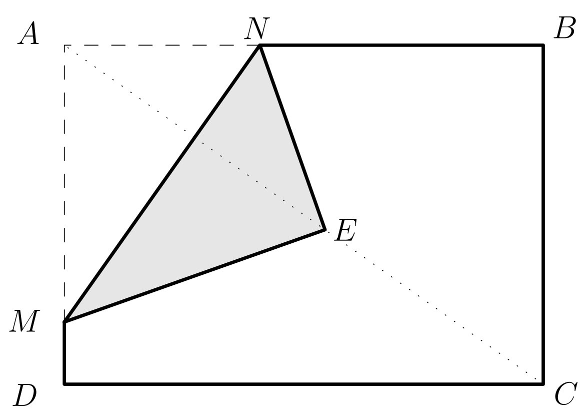
(c) Once again we try first to understand what we expect to happen. We know from (a) that the chance of my having four of a kind is very small. That is, even before I look at my cards, I feel fairly certain that I don't have "fours"; if I make this *absolutely* certain by looking at my cards, then I have scarcely changed the situation at all. So we should expect that the probabilities in (a) and (c) should be very similar, and we'll need to calculate in order to find out which is actually larger.

Working out the probability here is harder than in (a), and we'll do it by counting the number of possibilities for both my hand and my opponent's. The probability is equal to the number of deals in which he has "fours" and I do not, divided by the number of deals in which I do not (and he may or may not). For the denominator, I can have any hand ($C(52,5)$ possibilities) except for "fours" (13×48 possibilities); my opponent can then have any five cards chosen from 47, giving $C(47,5)$ possibilities. The numerator is the number of deals in which he has fours and I have anything ($13 \times 48\, C(47,5)$ possibilities), minus the number in which we both have fours. To choose a deal in this last category, choose a "four" for my opponent (13 possibilities) and then one other card (48 possibilities); then a four for me (11 possibilities – two have been knocked out) and one final card (43 possibilities). Putting all this together, the probability that my opponent has "four of a kind", if it is known that I do not, is

$$\frac{13 \times 48\, C(47,5) - 13 \times 48 \times 11 \times 43}{(C(52,5) - 13 \times 48)\, C(47,5)} = 0{\cdot}0002400796454\ .$$

This is less than the probability in (a), though, as we predicted, the difference is very small.

Q1405 A sheet of A4 paper has proportions $1 : \sqrt{2}$. Suppose such a sheet is grey on one side and white on the other, and label the corners A, B, C, D. The sheet is laid white side up and corner A is folded over, as shown in the diagram, in such a way that A, E, C are collinear and the length of the crease MN is equal to that of the side AD. What proportion of the visible figure $MNBCD$ is grey?



(And another question: **why** does a sheet of A4 paper have proportions $1 : \sqrt{2}$?)

SOLUTION Let O be the intersection of AC and MN. The triangles $\triangle ABC$ and $\triangle AON$ and $\triangle MAN$ are all right–angled. The first two have $\angle BAC$ in common, so they are similar; the last two have $\angle ANM$ in common, so they are similar; thus $\triangle ABC$ and

$\triangle MAN$ are similar. By Pythagoras' Theorem we calculate the ratio of hypotenuses

$$\frac{MN}{AC} = \frac{AD}{AC} = \frac{1}{\sqrt{3}}\ ;$$

therefore the ratio of areas is

$$\frac{\triangle MAN}{\triangle ABC} = \frac{1}{3}\ .$$

This means that $\triangle MAN$ has area which is $\frac{1}{6}$ of the whole sheet $ABCD$; but $\triangle MEN$ has the same area, and so the proportion we are looking for is

$$\frac{1/6}{5/6} = \frac{1}{5}\ .$$

The reason that an A4 sheet has proportions $1 : \sqrt{2}$ is that then half of the sheet has the same proportions (easy exercise: prove it), which means that two A4 sheets can be reduced and printed on one A4 sheet without being distorted.

Correct solution (using methods of coordinate geometry) received from an anonymous correspondent.

Q1406 Let n be a positive integer. Find polynomials $f(x)$ and $g(x)$ such that the only coefficients of these polynomials are 1 and -1, and

$$\frac{f(x)}{g(x)} = x^n - 2x^{n-1} + 1\ .$$

SOLUTION We have

$$\begin{aligned} x^n - 2x^{n-1} + 1 &= (x^{n-1} - x^{n-2} - \cdots - 1)(x-1) \\ &= \frac{(x^{n-1} - x^{n-2} - \cdots - 1)(x^n - 1)}{(x^n-1)/(x-1)}\ . \end{aligned}$$

The point of writing it this way is that when we multiply the factors in the numerator, there will be no "collecting" of terms and so all the coefficients will still be 1 or -1; while the quotient in the denominator is a well–known expression giving a polynomial with coefficients 1 only. That is,

$$x^n - 2x^{n-1} + 1 = \frac{x^{2n-1} - x^{2n-2} - \cdots - x^{n-1} + x^{n-2} + \cdots + 1}{x^{n-1} + x^{n-2} + \cdots + 1}\ ,$$

and, as required, all the coefficients in both the numerator and the denominator of the fraction are 1 or -1.

NOW TRY problem 1420.

Q1407 Extend the game of Chomp (see Michael Deakin's article in volume 48, number 3) to a three–dimensional array: whenever an asterisk is removed, so are all those which are to the north and/or east and/or higher than that one. The loser is the player who removes the asterisk in position $(1, 1, 1)$. If Alice, playing Bob on a $2 \times 2 \times 2$ grid, is the one to move first, find all her winning moves (if any).

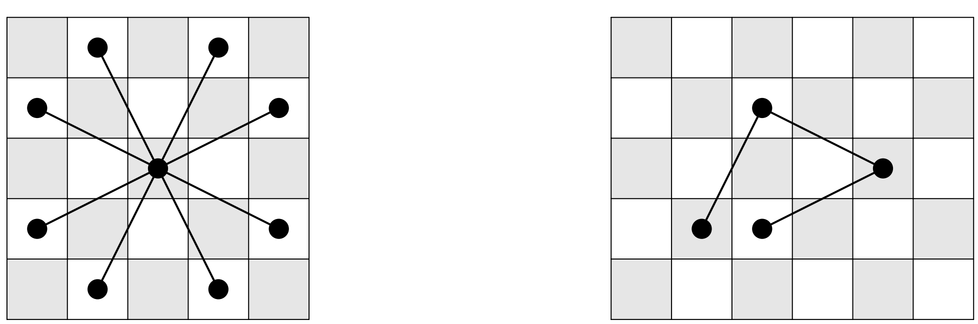
SOLUTION First notice that because of the symmetry of the grid, the initial moves $(1, 1, 2)$, $(1, 2, 1)$ and $(2, 1, 1)$ are, in effect, all the same; so are $(2, 2, 1)$, $(2, 1, 2)$ and $(1, 2, 2)$. Therefore we only really need to consider four cases.

- Taking $(1, 1, 1)$ obviously loses immediately.
- If Alice begins by taking $(1, 1, 2)$, she removes a whole layer of the grid and turns it into a two–dimensional game on a 2×2 grid. Bob has the first move in this "sub–game" and therefore Alice loses.
- If Alice takes $(2, 2, 1)$, which also removes $(2, 2, 2)$, then Bob can reply with $(1, 1, 2)$, taking with it $(1, 2, 2)$ and $(2, 1, 2)$. This leaves an "L–shape" of three asterisks (draw a diagram if you have trouble seeing this). It is Alice's turn to play and it's not hard to see that she must lose.
- If Alice begins with $(2, 2, 2)$, then Bob's reply must be, in effect, one of the three options we have already considered for Alice; therefore Bob loses and Alice wins.

Therefore Alice's only winning move is $(2, 2, 2)$.

Comment. From the argument given in Michael Deakin's article, we know that Alice does in fact have a winning move. Michael only dealt with the two–dimensional game, but exactly the same reasoning applies to the three–dimensional case. So once we have seen that every move other than $(2, 2, 2)$ loses for Alice, we don't actually need any further argument to know that $(2, 2, 2)$ wins!

Q1408 In the game of chess, a knight moves two squares horizontally or vertically and then one square in a perpendicular direction. So, as long as the edges of the board



(a) (b)

do not get in the way, the knight has at any stage eight possible moves, as shown in the first diagram. From the second diagram we can see that it is possible for a knight (making more than one move) to move from one square to that immediately to the right. Now suppose that a "superknight" moves in a similar fashion but with three steps in one direction and eight in a perpendicular direction. Also suppose that the board is infinitely large, so that there are no edges to get in the way. Can a superknight move from one square to that immediately to the right?

SOLUTION We shall write the superknight's move as $(3, 8)$, meaning 3 squares to the right and 8 upwards. The knight can also make moves of $(8, 3)$, $(-3, 8)$ and so on. We wish either to add up lots of these moves to get a result of $(1, 0)$, or to prove that it's not possible to do so. We'll begin by just getting a little closer to the start point: we have

$$(8, -3) + (-8, -3) + (3, 8) = (3, 2)$$

– that is, in plain language: 8 right 3 down, followed by 8 left 3 down, followed by 3 right 8 up, takes the knight altogether 3 squares right and 2 squares up from the starting point. Repeating the same idea with $(3, 2)$ and related moves, we have

$$(3, 2) + (-3, 2) + (2, -3) = (2, 1)$$

– we have now "synthesised" an ordinary knight's move – and then

$$(2, 1) + (-2, 1) + (1, -2) = (1, 0) \ .$$

We have achieved our object, but it could probably do with a bit of clarification. If we take the second equation above and change all upwards moves to downwards and vice versa, we have the related equation

$$(-3, 2) + (3, 2) + (-2, -3) = (-2, 1) \ ;$$

if we now interchange rightwards and upwards moves, we obtain

$$(2, -3) + (2, 3) + (-3, -2) = (1, -2) \ .$$

We can substitute all these into the third equation: moves such as $(3, 2)$ and $(-3, -2)$ are just the opposites of each other and can be cancelled. This gives

$$(3, 2) + 2(-3, 2) + 2(2, -3) = (1, 0) \ .$$

Using the first equation in a similar way, we have

$$(-8, -3) + (8, -3) + (-3, 8) = (-3, 2)$$

and

$$(-3, -8) + (-3, 8) + (8, -3) = (2, -3)$$

and so, finally,

$$5(8, -3) + 3(-8, -3) + (-3, -8) + 4(-3, 8) = (1, 0) \ .$$

That is, our "superknight" can get one square to its right in thirteen moves: five of 8 right 3 down, three of 8 left 3 down, one of 3 left 8 down and four of 3 left 8 up.

NOW TRY problem 1414.

Q1409 Suppose that the following statement (related to the ABC Conjecture – see the editorial in the previous issue) is true:

If a, b, c are positive integers such that a, b have no common factor and $a + b = c$; and if P is the product of all prime factors, taken once each, of a, b, c; then $c < P^2$.

Prove the following result, which is (almost) Fermat's Last Theorem:

There are no positive integers x, y, z and n, with $n > 5$, such that $x^n + y^n = z^n$.

SOLUTION We shall prove the result by contradiction: that is, we assume that

$$x^n + y^n = z^n \,, \tag{1}$$

where x, y, z, n are positive integers and $n > 5$, and we shall show that this leads to an impossible conclusion. First of all, if x and y have a common factor g, then g is also a factor of z, and so both sides of (1) can be divided by g^n to obtain another solution. We may therefore assume that this has already been done, and that x, y have no common factor. Now let $a = x^n$ and $b = y^n$ and $c = z^n$; then a, b have no common factor and $a + b = c$, so we may apply the assumed result to conclude that

$$z^n < P^2 \,.$$

However the prime factors of x^n, y^n, z^n are exactly the same as those of x, y, z, and therefore

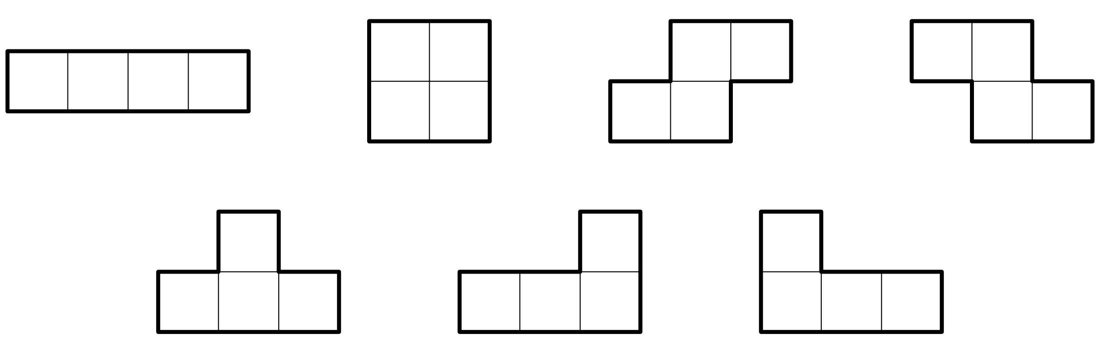
$$P = xyz \,;$$

therefore we have $z^n < x^2y^2z^2$. But from (1) it is clear that $x < z$ and $y < z$, so finally we get

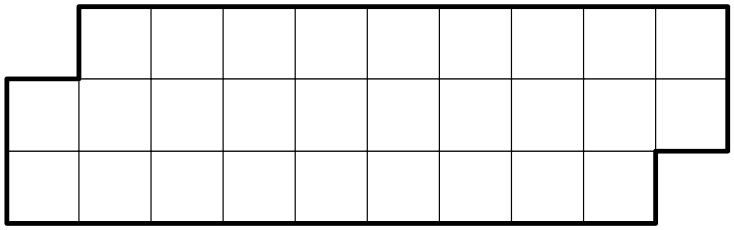
$$z^n < z^6$$

which is impossible since $n \geq 6$. Therefore (if the assumed result is true) there are no positive integers x, y, z and n, with $n > 5$, such that $x^n + y^n = z^n$.

Q1410 A **tetromino** is a shape constructed by joining four squares of unit size along their edges. There are seven such shapes (which will be familiar to anyone who has ever played Tetris) and they are shown below.



The shapes obviously have a total area of 28. Can they be arranged so as to completely cover the following shape of area 28?



SOLUTION Colour the large region with alternating black and white squares, in chessboard fashion. Each of the tetrominos, wherever it is placed on this region, will cover two white squares and two black, except for the "T–shaped" tetromino, which will cover three of one colour and one of the other. Altogether, they will cover 15 squares of one colour and 13 of the other. But the large shape has 14 squares of each colour, and therefore cannot be covered exactly by the seven tetrominos.