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Solutions 1411–1420

Q1411 How many three–digit numbers are there which do not contain any digit more than once? What do you get if you add them all up?

SOLUTION There are 9 possible choices for the first digit (it can't be zero); then 9 for the second digit (it can't be the same as the first) and 8 for the third. Altogether there are $9 \times 9 \times 8 = 648$ such numbers.

If the first digit is 1 then there are, as above, $9 \times 8 = 72$ possibilities for the remaining digits; thus, in our collection of numbers, a 1 occurs in the hundreds place 72 times. The same is true for every other non–zero digit, so the total of all the digits in the hundreds places is

$$72 \times 100 + 72 \times 200 + \cdots + 72 \times 900 = 324000 .$$

For any specific non–zero digit in the tens place there are 8 possibilities for the hundreds digit and 8 for the units digit, 64 altogether. So the total of all the digits in the tens places is

$$64 \times 10 + 64 \times 20 + \cdots + 64 \times 90 = 28800$$

(we haven't counted the zeros in the tens places, but obviously they don't add anything to the sum). Likewise, the total of all the digits in the units places is

$$64 \times 1 + 64 \times 2 + \cdots + 64 \times 9 = 2880 ,$$

and the sum of all the numbers is 355680.

Q1412 Find all solutions of the equation

$$x^4 + 4x^3 - 6x^2 - 20x + 13 = 0 .$$

SOLUTION Completing the square (in a slightly unusual way), we can write the equation as

$$(x^2 + 2x)^2 - 10(x^2 + 2x) + 13 = 0 .$$

If we write $q = x^2 + 2x$, this equation is the quadratic $q^2 - 10q + 13 = 0$, which has solutions $q = 5 \pm \sqrt{12}$. To complete the problem we have to solve two further quadratics

$$x^2 + 2x = 5 + \sqrt{12} \quad \text{and} \quad x^2 + 2x = 5 - \sqrt{12} ,$$

and there are altogether four solutions

$$x = -1 + \sqrt{6 + \sqrt{12}} , \quad x = -1 + \sqrt{6 - \sqrt{12}} ,$$
$$x = -1 - \sqrt{6 + \sqrt{12}} , \quad x = -1 - \sqrt{6 - \sqrt{12}} .$$

NOW TRY problem 1429.

Q1413 By arranging the digits of the number 2013 we obtain twenty–four different numbers 0123, 0132, ..., 2013, ..., 3210, where the first digit of a number **is** permitted to be zero. Find the smallest possible value of

$$|x - 0123| + |x - 0132| + \cdots + |x - 2013| + \cdots + |x - 3210| ,$$

where x is a real number.

SOLUTION Given any n different real numbers $x_1, x_2, \ldots, x_n$, with $x_1 < x_2 < \cdots < x_n$, we write

$$S(x) = |x - x_1| + |x - x_2| + \cdots + |x - x_n|$$

for any real number x. Our aim is to find the minimum value of $S(x)$, if $x_1, x_2, \ldots, x_n$ are the numbers stated in the question. We begin by sketching the graph of $y = S(x)$. If $x_k \le x \le x_{k+1}$ then we have

$$\begin{aligned} y &= (x - x_1) + \cdots + (x - x_k) + (x_{k+1} - x) + \cdots + (x_n - x) \\ &= (kx - x_1 - \cdots - x_k) + (x_{k+1} + \cdots + x_n - (n-k)x) \\ &= (2k - n)x - (x_1 + \cdots + x_k - x_{k+1} - \cdots - x_n) . \qquad (*) \end{aligned}$$

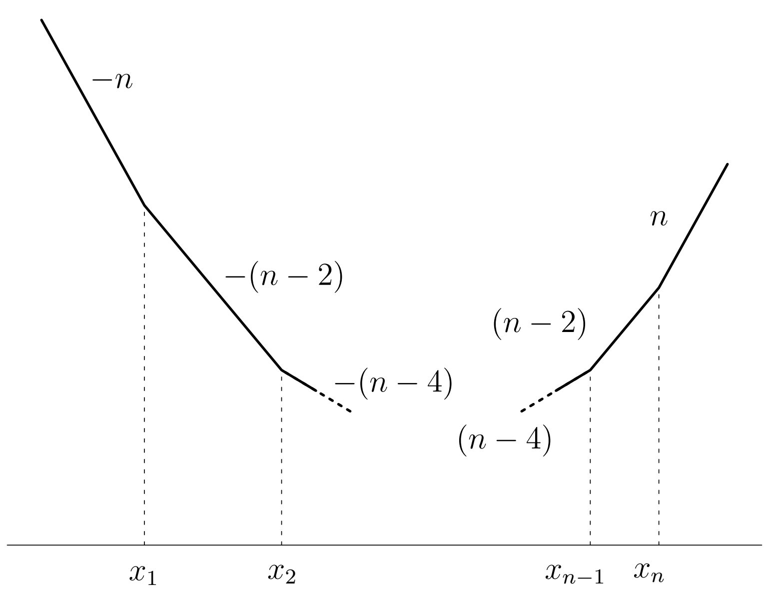
Since n and k and all the x_j are fixed numbers, this is just the equation of (part of) a straight line with gradient $2k - n$. If $x < x_1$ then

$$y = (x_1 - x) + \cdots + (x_n - x) = -nx + (x_1 + \cdots + x_n) ,$$

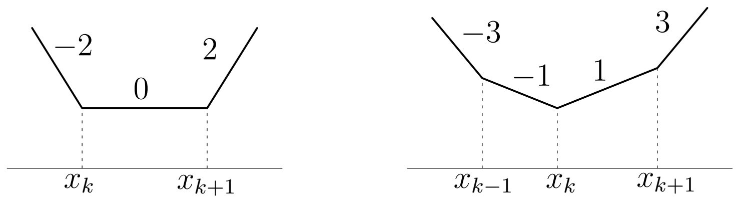
a line of gradient $-n$, while if $x > x_n$ then

$$y = (x - x_1) + \cdots + (x - x_n) = nx - (x_1 + \cdots + x_n) ,$$

a line of gradient n. So the graph of $y = S(x)$ will be as shown in the following diagram, where we have labelled each line segment with its gradient.



The minimum value of $S(x)$ will occur at the "bottom" of the graph, which will look like one of the following diagrams,



the first if $n = 2k$ is even and the second if $n = 2k - 1$ is odd. (Again each segment is labelled with its gradient.) For the given question we have $n = 24 = 2 \times 12$ and so the minimum value of $S(x)$ occurs for any value of x between x_{12} and x_{13}; this minimum value is given by the expression $(*)$ with $n = 24$ and $k = 12$, and is

$$S(x) = -0123 - \cdots - 1320 + 2013 + \cdots + 3210 .$$

You can just add up these numbers, or, more conveniently, you can use the ideas from the solution to problem 1411: in either case you should find that the minimum value is $S(x) = 23112$.

NOW TRY problem 1424.

Q1414 Let a and b be positive integers, and consider a "knight–like" piece which moves on a chessboard a squares up, down, left or right and then b squares in a perpendicular direction. We shall refer to such a piece as an (a, b)–*superknight*: for example, the piece described in problem 1408 is a $(3, 8)$–superknight, while a $(2, 1)$–superknight is just an ordinary knight.

Find all values of a and b for which an (a, b)–superknight can move from one square to that immediately to the right. As in problem 1408, we assume that the board is infinitely large, so that there are no edges to get in the way.

SOLUTION For this to be possible, a and b must have no common factor greater than 1: for if they have a common factor $g > 1$, then every move involves either a or b steps to the right or left; so the superknight must always be a multiple of g squares to the right or left of its initial square, and cannot be one square to the right. Furthermore, a and b must not both be odd: for if this is so, then any move by the superknight takes it from one chessboard square to another of the same colour, and it can never reach the square immediately to its right, which has the opposite colour.

There are no further requirements on a and b. To prove this, we shall show that if a and b are positive integers, not both odd and having no common factor, then an (a, b)–superknight can move from one square to that immediately to the right. The proof will imitate the case of the $(3, 8)$–superknight in problem 1408. So, suppose that we have an (a, b)–superknight; since this is the same as a (b, a)–superknight, we may assume that $0 < b < a$. We can make a sequence of moves (see solution 1408 for an explanation of the notation)

$$(a, -b) + (-a, -b) + (b, a) = (b, c) , \quad \text{where} \quad c = a - 2b ;$$

by turning all the moves "upside down" we may assume that $c \geq 0$. Now if $c = 0$ then $a = 2b$; since a and b have no common factor, this is only possible when $b = 1$; so we have reached $(1, 0)$ and we are finished. If, on the other hand, $c > 0$, then we have "synthesised" the move of a (b, c)–superknight. Now b and c have no common factor greater than 1 (if they did, it would also be a common factor of a and b); and they are not both odd (if they were, so would be a and b). Moreover, b and c are both less than a. So b and c satisfy the same conditions as a and b; and the (b, c)–superknight is "smaller" than the original (a, b)–superknight; therefore we can repeat the procedure, as we did in the solution to problem 1408, obtaining superknights with smaller and smaller moves, until at last we reach the smallest possible superknight, which is an ordinary knight. And we know that we can use this to get from one square to that immediately to the right, so we are finished.

NOW TRY problem 1425.

Q1415 How often is the sun directly overhead at the equator? Once a day? Twice a day? Something else? Explain your answer!

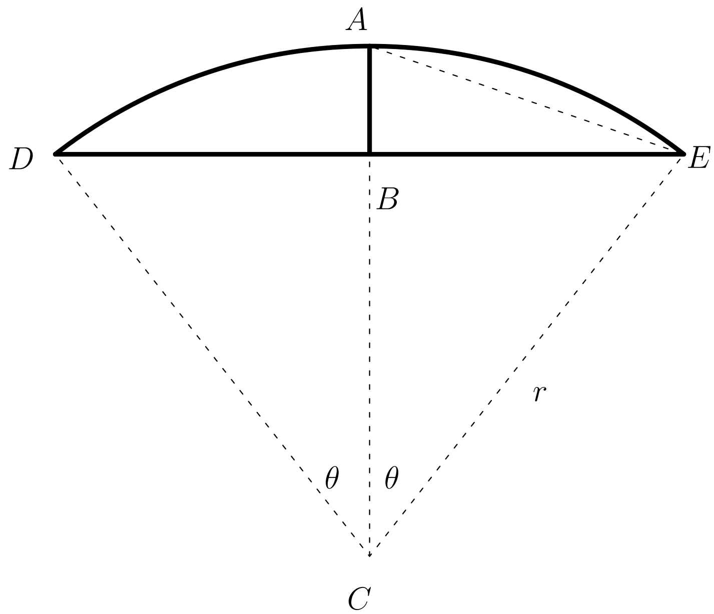
SOLUTION This is a question of "applied" rather than of "pure" geometry, and to answer it we will of course need to know some basic facts about the solar system – fortunately, not too many. The earth revolves around the sun once per year, tracing an elliptical path in a plane which also includes the sun – this is known as the *plane of the ecliptic*. At the same time, the earth rotates about its axis, the line passing through the north and south poles. Imagine a plane passing through the earth's equator and extending indefinitely in every direction. This *equatorial plane* rotates with the earth, and so its position as a whole, relative to the earth, never changes. As the earth travels in its orbit around the sun, the polar axis and the equatorial plane move with it, always retaining the same orientation. The sun is directly overhead at the equator whenever it is found to be on the equatorial plane. If the polar axis were perpendicular to the ecliptic then the sun would always be on the equatorial plane; in reality, however (and this is the last astronomical fact we shall need), the axis is tilted away from the ecliptic, and the equatorial plane is also tilted away from the ecliptic plane. As the earth revolves once about the sun (here you will probably wish to draw some diagrams or wave your hands a bit – imagine the sun as fixed and the equatorial plane moving around it), the equatorial plane will contain the sun twice, once "on each side" of the earth's orbit.

In conclusion, the sun is directly overhead at the equator twice a year.

Q1416 A chord cuts off an arc of a circle. If the chord length is c, the arc length s and the maximum (perpendicular) distance from the chord to the arc h, prove that

$$\frac{2h}{c} = \tan\frac{2hs}{c^2 + 4h^2} \ .$$

SOLUTION Let r be the radius of the circle and 2θ the angle subtended by the chord (and arc).



In the diagram, C is the centre of the circle and AC is a radius perpendicular to the chord. We use two theorems of circle geometry:

- the angle at the circumference subtended by an arc is half the angle at the centre subtended by the same arc;
- a diameter perpendicular to a chord bisects the chord.

The first of these, applied to the arc AD, shows that $\angle AEB = \frac{1}{2}\theta$, the second that the length of BE is $\frac{1}{2}c$; and by definition the length of AB is h. Therefore

$$\frac{h}{c/2} = \tan\frac{\theta}{2}\ .$$

On the other hand, by using the length of an arc formula, and Pythagoras' theorem in $\triangle BCE$, we have

$$s = 2r\theta \quad\text{and}\quad r^2 = (r-h)^2 + \left(\frac{c}{2}\right)^2\ .$$

Eliminating θ by means of the first of these, and solving for r in the second, we obtain

$$\frac{2h}{c} = \tan\frac{s}{4r} = \tan\frac{2hs}{c^2+4h^2}$$

as claimed.

Q1417 If n is a positive integer, let

$$P = 1^1 \times 2^2 \times 3^3 \times \cdots \times n^n\ .$$

Without using a calculator, prove that if $n = 17$ then P is larger than a googol. (Remember that a googol is the number consisting of a 1 followed by a hundred 0s, that is,

10^{100}.) For a harder question, prove that in fact if $n = 15$ then P is already larger than a googol – still without using a calculator of course!

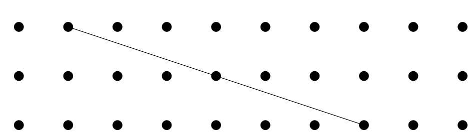
SOLUTION If $n = 17$ we can find a very simple lower estimate for P by just ignoring many of the terms in the product:

$$\begin{aligned} P &= 1^1 \times \cdots \times 10^{10} \times 11^{11} \times \cdots \times 17^{17} \\ &> 10^{10} \times 11^{11} \times \cdots \times 17^{17} \\ &> 10^{10} \times 10^{11} \times \cdots \times 10^{17} \\ &= 10^{10+11+\cdots+17} \\ &= 10^{108} \end{aligned}$$

which is bigger than a googol. For $n = 15$ this approach won't work (try it!) and we have to be a bit more clever. We'll group together terms like 7×15 and 8×13, because they are just slightly bigger than 100. We have

$$\begin{aligned} P &= 1^1 \times \cdots \times 10^{10} \times 11^{11} \times \cdots \times 15^{15} \\ &> (7 \times 15)^7 \times 15^8 \times 14^{14} \times (8 \times 13)^8 \times 13^5 \\ &\qquad \times (9 \times 12)^9 \times 12^3 \times 11^{11} \times 10^{10} \times (2 \times 5) \\ &> (10^2)^7 \times 10^8 \times 10^{14} \times (10^2)^8 \times 10^5 \\ &\qquad \times (10^2)^9 \times 10^3 \times 10^{11} \times 10^{10} \times 10 \\ &= 10^{100}\ . \end{aligned}$$

Q1418 Consider a square lattice of points consisting of m rows and n columns. In how many different ways is it possible to choose m points, one from each row, which are collinear? The case $m = 3$, $n = 10$ is illustrated, together with one possible set of three points.



SOLUTION Suppose that we have a set of collinear points, one from each row. If the point in row 1 lies in column i and that in row 2 lies in column $i + k$, then the one in row 3 must lie in column $i + 2k$, and so on, the last point being in row m and column $j = i + (m - 1)k$. Thus any satisfactory selection must include points in row 1, column i and in row m, column j, where $i - j$ is a multiple of $m - 1$. Conversely, if we choose two such points, then the line joining them will include a point in every row and will be a permissible selection.

So, we have to count the number of pairs (i, j), where i and j are integers from 1 to n which have the same remainder when divided by $m - 1$. Note that i and j may be the same; and if they are different it does not matter which one is larger: *any* pair of numbers having the same remainder will do. Divide n by $m-1$ to give a quotient q and remainder r, that is,

$$n = q(m-1) + r \quad \text{with} \quad 0 \le r < n-1 \, . \qquad (*)$$

We can think of the numbers from 1 to n as being split up into $m-1$ sets of q numbers, each having the same remainder when divided by n, with the first r of these sets having an extra element. That is, we have r sets containing $q+1$ numbers, and $m-1-r$ sets containing q numbers, and we wish to know in how many ways we can choose two numbers which are both in the same one of these sets. Thus the answer to the problem is

$$r(q+1)^2 + (m-1-r)q^2 \, .$$

Although this formula is not written in terms of the given numbers m and n, if they are known then q and r can be determined from equation $(*)$; therefore this is a satisfactory solution.

Q1419 Given an integer $n \ge 2$, what is the greatest number that can be obtained by writing n as a sum of positive integers and multiplying those integers? For example, if $n = 2013$ we could write $n = 1006 + 1007$ and obtain the product $1006 \times 1007 = 1013042$, or $n = 1000 + 1000 + 13$ giving $1000 \times 1000 \times 13 = 13000000$, but neither of these is the maximum possible.

SOLUTION In order to obtain the maximum product, none of the summands (the numbers in the sum making up n) can be 1; for any 1 could be combined with another summand to increase the product. For example, replacing $1+2$ by 3 increases the product from 2 to 3. Furthermore, none of the summands can be 5 or more, as 5 could be replaced by $2+3$, and 6 could be replaced by $2+4$, and so on, in each case increasing the product of summands. So to get the largest possible product, we should write n as a sum of 2s, 3s and 4s only. And indeed, a summand of 4, while permissible, is not needed, because 4 could be replaced by $2+2$, leaving the product unchanged; so we may write n as a sum of 2s and 3s only. Finally, we must not have three or more 2s, since replacing $2+2+2$ by $3+3$ will again give a larger product, 9 instead of 8.

Therefore, we shall use 2s and 3s only, with no more than two 2s. And since this leaves only one way to create a sum adding up to any given number, the problem is finished. Specifically, if we divide n by 3 we shall obtain a remainder of $0, 1$ or 2:

- if the remainder is 0 then we write $n = 3 + \cdots + 3$, giving a maximal product of $3^{n/3}$;
- if the remainder is 1 then we write $n = 2+2+3+\cdots+3$ with $(n-4)/3$ summands of 3, giving a product $4 \times 3^{(n-4)/3}$;
- if the remainder is 2 then we write $n = 2+3+\cdots+3$, giving a product $2 \times 3^{(n-2)/3}$.

Comment. In the case of remainder 1 we could alternatively have taken $n = 4 + 3 + \cdots + 3$, resulting in the same product $4 \times 3^{(n-4)/3}$.

NOW TRY problem 1427.

Q1420 Let n be a positive integer. Show that if n is odd, then it is **not possible** to find polynomials $f(x)$ and $g(x)$ such that the only coefficients of these polynomials are 1 and -1, and

$$\frac{f(x)}{g(x)} = x^n - x^{n-1} + 1 \ .$$

Note that in particular, $f(x)$ and $g(x)$ may not have any zero coefficients: for example $g(x) = x^3 - x + 1$ does not meet the requirements of the question.

SOLUTION If $f(x)$ is a polynomial having coefficients 1 and -1 only and degree m, then $f(1)$ is a sum of $m+1$ terms, each equal to 1 or -1; that is, it is a sum of $m+1$ odd numbers. Thus, $f(1)$ is even if m is odd, and odd if m is even. Similarly, if $g(x)$ has coefficients 1 and -1 only and degree k, then $g(1)$ is even if if k is odd and odd if k is even. Now suppose that $f(x)$ and $g(x)$ also satisfy the given equation. Then by substituting $x = 1$ and also comparing degrees, we have

$$f(1) = g(1) \quad \text{and} \quad m = n + k \ .$$

However, if n is odd then these equations are inconsistent with the previous conclusions. For the first equation implies that m and k are both odd or both even, while the second implies that one is odd and the other even. The situation is impossible, and we must conclude that there are no polynomials satisfying the requirements of the question.