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History of Mathematics: Proving infinitely many things for the price of two

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One of the most used methods of proof applies to those situations in which an infinite set of propositions follow one another in a sequence. An example that is very straightforward mathematically but historically important states that the sum of the first n odd integers will always be n^2. In symbols:

$$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2. \tag{0.1}$$

This simple formula involves, in fact summarizes, infinitely many assertions. We are saying that

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \\ 1+3+5+7+9 &= 5^2 \\ &\;\;\cdots\cdots \end{aligned}$$

Here is one approach to the proof of these assertions:

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                                          + + + + +
                              + + + +     * * * * +
                    + + +     * * * +     * * * * +
          + +       * * +     * * * +     * * * * +
    +     * +       * * +     * * * +     * * * * +
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The first five instances of the formula correspond exactly with the first five of these (potentially infinitely many) diagrams. Each diagram (with the partial exception of the first) consists of a square array of asterisks together with an "outer edge" of plus signs. In the fourth diagram, for example, there are $(4-1)^2$ $(= 9)$ asterisks and 7 $(= 2 \times 4 - 1)$ plus signs. So the total number of entries (asterisks and plus signs) is $9 + 7 = 16 = 4^2$ in accordance with the formula. We see here an underlying pattern, and it is not difficult to envisage this as persisting forever.

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However, the mere recognition of a pattern is not of itself adequate as proof. Here it *is* possible to construct a valid proof from the recognition of the underlying pattern, although this is not the route I will take as something better is available. But before getting onto this, let me embark on a cautionary tale.

Imagine a circle, and place two points on its circumference. Join these points. We now find that the interior of the circle is divided into two distinct regions. Now place a third point on the circumference, and join this to the other two. We now discover that the interior of the circle is partitioned into four distinct regions. Continue in this way, each time adding a new point and joining it to all the previous ones. If we denote by n the number of points and by $f(n)$ the number of regions, we find

$$f(1) = 1, \quad f(2) = 2, \quad f(3) = 4, \quad f(4) = 8, \quad f(5) = 16.$$

(Here I have interpolated the first, trivial, entry and have left to the reader the tedious, but otherwise straightforward, task of verifying the last two.) Surely, we have here a general rule $f(n) = 2^{n-1}$. This guess is extremely plausible, but unfortunately is also wrong! Again I leave it to the reader, but with patience one can discover that $f(6) = 31$, whereas we would have guessed 32. In fact there *is* a general rule: it is

$$f(n) = \frac{1}{24}\left[n^4 - 6n^3 + 23n^2 - 18n + 24\right]$$

I won't stop to prove this result here, but interested readers will find (essentially) this problem discussed at length by Ross Honsberger in his book *Mathematical Gems* (pp. 99-107).[2] My purpose here is to show that proof is required; we can't simply rely on a pattern that we think we have found.

Equation (1) was, as indicated above, really an infinity of statements, all of which are claimed to be true. If we write $S(n)$ as a shorthand for the statement "Statement S about the number n is true", then all that we need do to prove $S(n)$ for all values of n is to show two simpler things:

(1). $S(1)$,

(2). $S(n) \Rightarrow S(n+1)$, for all n.

The reason is as follows: if, by (1), $S(1)$ holds, then, by (2), $S(2)$ holds, and then, again by (2), $S(3)$ holds, and so on for all values of n.

So let us apply this method to Equation (1).

Clearly $S(1)$ is true as $1 = 1^2$.

Now suppose that for some subsequent value of n, $S(n)$ holds, i.e. for this particular value of n,

$$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2 \qquad (0.2)$$

Then we need to show that this would imply that

[2] Actually, Honsberger deduces a slightly different result. If we dente by $g(n)$ the formula he gives, we find that $f(n) = g(n) + n$.

$$1+3+5+7+....+(2n-1)+(2n+1)=(n+1)^2.$$

But because we have assumed that $1+3+5+7+....+(2n-1)=n^2$, then all that remains to show is that $n^2+(2n+1)=(n+1)^2$, and this is just a piece of very simple algebra.

I deal with this case in detail because it is historically important and also raises a number of methodological questions. Equation (1) was first given explicitly by the Sicilian mathematician Francisco Maurolico[3] (1494-1575) in his book *Arithmeticorum libri duo,* written in 1557. Here he asserts and proves that the sum of the n^{th} square number and the $(n+1)^{st}$ odd number is the $(n+1)^{st}$ square number, much as I have just given. To put the proof in this form removes any lingering doubts we might have concerning the generality of the underlying pattern.

Also, this case is important because for probably a hundred years or so, it was believed that Maurolico's use of this type of argument was the first. It goes by the name of "mathematical induction"[4] and is a most useful *method* of proof, one of a quite small set of such. It was an Italian Mathematician called Vacca who first pointed out Maurolico's rôle in the story. Vacca had published this discovery, but in a rather obscure Italian journal, so that it was overlooked. When later, the eminent historian of mathematics Florian Cajori attributed the technique to the later mathematician Blaise Pascal (1623-1662), Vacca's contribution was pointed out to him. This led to some correspondence between Cajori and Vacca, and as a result Cajori saw to it that a letter from Vacca was published in the *Bulletin of the American Mathematical Society*. This drew the attention of the wider mathematical community to Maurolico's contribution.[5] It does remain true, however, that Pascal gave a much clearer exposition of the method. Other, later, mathematicians have also been credited with the independent discovery of the method. Among these are the Jakob I Bernoulli (1654-1705) and Augustus De Morgan (1806-1871).

But this leaves open the question as to whether the technique was known or used before Maurolico. In particular, there have been from time to time suggestions that the ancient Greeks used it. Such suggestions even surface in some of the websites devoted to the topic. So some authors profess to find inductive proofs in Parmenides, Plato or Euclid. These suggestions were examined critically by the Israeli mathematician Sabetai Unguru in a paper published in 1991 in the Italian journal *Physis*, devoted to the history of science. Unguru examined such claims and dismissed them, showing

[3]There are many different spellings of his name, but this is that most usually given.

[4]The name is actually something of a misnomer. More generally, *induction* is the process of arguing from special cases to a general rule. It is opposed to *deduction*, which is the process of arguing from a general rule to particular cases. The entire scientific method relies on inductive reasoning; nevertheless in strict logic, induction is not necessarily valid, whereas deduction is. *Mathematical induction* is actually a form of *deduction* but the name is now standard, and has a certain aptness because it does prove a general rule by means of a special case and a rule of persistence (equivalent to a proof that the special case is typical).

[5]Not everyone accepts the Mauriloco claim, although Cajori did. If you set the bar very high and insist on a very precise statement of the underlying principle, then perhaps his enunciation of the method is still lacking. To my mind, however, this is nit-picking!

that the alleged uses of the method did not in fact have this status. As his abstract states

> "[My] article examines the alleged instances of mathematical induction in Greek mathematics showing them to be actually non-inductive."

Had this been the extent of Unguru's claim, it would have been a useful but not very important contribution, but Unguru wanted more. His abstract continued:

> "Furthermore the author argues for the *impossibility* of genuine inductive proofs within the confines of Greek mathematics."

Much of the article thus argues that the Greek mindset precluded the use of mathematical induction. This was a bold claim, and a much more difficult one to support. In particular, it aroused the ire of David Fowler, a specialist scholar of Greek mathematics. His reply was titled "Could the Greeks have used mathematical induction? Did they use it?". Fowler and Unguru adopted different ideas of what might constitute an inductive proof, Fowler taking a wider view, Unguru a narrower. Thus Fowler would have accepted the geometric approach I gave to Equation (1) as genuinely inductive, because the underlying pattern can quite easily be made watertight, and so the proof above reproach. But not only did this not satisfy Unguru's more stringent demand, but it failed to address the second of Fowler's questions "Did they use [induction]?".

This latter question however has since been answered, and shortly after the exchange of views I have just reported. That took place in 1994, and the following year a short note by a Melbourne schoolteacher, Hussein Tahir, drew attention to a passage in the works of the late Greek mathematician Pappus of Alexandria (c.290-c.350). It is a clear example of an inductive argument.[6] It concerns a geometric arrangement of circles, and it is too specialized to go into here, but now Fowler's question has been answered in the affirmative, and Unguru's position thus demolished; it also means that Maurolico was not (as had previously been thought) to be the first to use mathematical induction.

Other figures also find a place in the story of discovery. Perhaps the most prominent is the Islamic mathematician al-Karaji (953-c.1029). This part of the history is perhaps best described by Victor Katz in his book *A History of Mathematics: An Introduction*, a work which deliberately tries to overcome the eurocentricity that besets some accounts. al-Karaji set out to justify not Equation (1) but a somewhat related equation

$$\left(1^3 + 2^3 + 3^3 + ... + n^3\right) = \left(1 + 2 + 3 + ... + n\right)^2 . \tag{0.3}$$

He stated this result for the special case $n = 10$, and gives what Katz regards as an essentially inductive argument. This might satisfy Fowler's criterion but not Unguru's.

[6] Tahir published his note as a letter to the editor in the *Gazette of the Australian Mathematical Society* and it might have been overlooked as was Vacca's original paper on Maurolico, but in this case, the note was given a place in the journal of record *Mathematical Reviews*, so that it has been brought to the attention of the wider mathematical community.

The same comment applies to other claims for mathematicians in the years between Pappus and Maurolico.

But now I hear readers objecting:

> But surely this entire method of proof is unsound! When we set out to prove Equation (1), we did it by assuming the truth of Equation (2), which is exactly the same! This surely makes the whole argument circular! It isn't allowable to assume in the course of a proof the very thing that you are trying to prove!

The point is a good one and when I was still engaged in teaching, it was one which my students found a stumbling block. However, it does not invalidate either the proof of Equation (1) in particular nor the method of proof in general. The reason, however, is subtle. When we assert Equation (1) to be true, we mean that each case of it is true; that $1 = 1^2,\ 1+3 = 2^2,\ 1+3+5 = 3^2,\ 1+3+5+7 = 4^2, 1+3+5+7+9 = 5^2,$ and so on for ever and ever. Whatever value of n we choose, Equation (1) remains true.

When we come to Equation (2), which to be sure *looks the same*, it is actually *not*. Equation (2) applies only to a specific value of n, an unspecified one, to be sure, but by no means to *every* such value. Some authors use a device to make this clearer, representing the *particular* value of n by k, and so writing Equation (2) as

$$1 + 3 + 5 + 7 + ... + (2k - 1) = k^2. \tag{0.4}$$

To my mind, this does not address the student's source of confusion, because we would, in other contexts, regard Equation (4) as exactly the same as Equation (2). The key point is that the symbol n in Equation (1) applies to *all* values, whereas the symbol n (or k if you like) applies to a single (but unspecified) value. If calling it k helps to this end, then well and good, but the point is really the same.

What is at issue here is that the second part of the proof: $S(n) \Rightarrow S(n+1),$ for all n must be proved in general. What we must prove is that the formula applies when n is replaced by $n + 1$.[7]

Now let's look at some other examples. One of the examples that Fowler gave was the generation of the *triangular numbers*[8], These arise from the formula

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} \tag{0.5}$$

In order to do this, first note that the result holds true for $n = 1$. So now we need to deduce from Equation (5) that

[7] It used to raise my hackles when students introduced this part of the proof by "Let $n = n + 1$". However over the intervening years, several programming languages allow this usage. The meaning is "Replace n by $n + 1$", which is precisely what is intended here.

[8] So called because the numbers on the right-hand side of Equation (5) below can all be represented as triangular arrays.

$$1 + 2 + 3 + ... + n + (n + 1) = \frac{(n + 1)(n + 2)}{2}$$

This amounts to showing that

$$\frac{n(n + 1)}{2} + (n + 1) = \frac{(n + 1)(n + 2)}{2}$$

And I leave this part to the reader. And now that this is proved, I can also leave to the reader the task of proving Equation (3).

However, a certain amount of care is required as we use this technique. A cautionary tale is provided by the following notorious example.

It is proposed to show that **All Natural Numbers are Equal**.

The proof is inductive. Clearly, in a set of just one number all its members are equal. Now suppose that for some number n all the n members of a set of n numbers are equal. Now take a set of $n + 1$ numbers, the n of the original set together with a new one. From this set take a subset of n members, the new one and $n - 1$ of the others. This is a set of n numbers, all of which are thus equal. The new number must therefore have the same value as all the others, and so it follows that *all* numbers are equal.

The proof is of course fallacious, and I leave it to the reader to find where the error lies. This piece of instructive nonsense is attributed to the mathematician and educator George Pólya (1887-1985), who proposed a somewhat more lurid example, 'proving' that all horses are of the same color. It has also been recast to 'prove' that all girls have the same color eyes.

But enough of this frippery, and back to the serious stuff. The method of proof here outlined can be extended in various ways. Here is a more meaty example.

$$\left(\frac{n}{2}\right)^n > n! \text{ for } n \geq 6. \qquad (0.6)$$

A little experimentation with a calculator shows that the inequality is false for $n \leq 5$, so we cannot assert $S(1)$, but when $n = 6$, we find

$$\left(\tfrac{6}{2}\right)^6 = 729 > 720 = 6!,$$

So we start at $n = 6$. $S(6)$ is true. We now need to prove that, for values of n larger than 6,

$$\left(\tfrac{n}{2}\right)^n > n! \text{ implies } \left(\tfrac{n+1}{2}\right)^{n+1} > (n + 1)!$$

So, assume that there is a value of n for which $\left(\frac{n}{2}\right)^n > n!$, the *inductive hypothesis*. (We know there is at least one such n because it is true for $n = 6$.) In order to prove Equation (6), first prove

$$\left(1 + \tfrac{1}{n}\right)^n < \left(1 + \tfrac{1}{n+1}\right)^{n+1}$$

There are various ways to do this, but perhaps the simplest is to consider the ratio

$$\left(1+\frac{1}{n+1}\right)^{n+1} / \left(1+\frac{1}{n}\right)^{n}.$$

But this may be simplified to $\left(\frac{n+2}{n}\right)^{n}\left(\frac{n+2}{n+1}\right)$, a product of factors all greater than 1 and so itself greater than 1. It follows that as n increases, so does $\left(1+\frac{1}{n}\right)^{n}$ and so we are secure in the claim that $\left(1+\frac{1}{n}\right)^{n} > 2$, its value when $n = 1$. Call this the *subsidiary result*.

We are now ready for the inductive aspect of the proof. Suppose Inequality (6) holds for some value of n. Then for *this* value of n, $\left(\frac{n}{2}\right)^{n} > n!$. We need to show that, under this assumption, $\left(\frac{n+1}{2}\right)^{n+1} > (n+1)!$. But now

$$\begin{aligned}
\left(\tfrac{n+1}{2}\right)^{n+1} &= \left(\tfrac{n+1}{2}\right)^{n}\left(\tfrac{n+1}{2}\right) \\
&= \left(1+\tfrac{1}{n}\right)^{n}\left(\tfrac{n}{2}\right)^{n}\tfrac{n+1}{2} \\
&> 2\left(\tfrac{n}{2}\right)^{n}\tfrac{n+1}{2} && \text{by the subsidiary result just proved} \\
&> (n+1)n! && \text{by the inductive hypothesis} \\
&= (n+1)!
\end{aligned}$$

So now the proof is complete.

There is yet another variant that needs to be mentioned. In this, it is assumed that a result is known to hold true for all values of n up to and including some number N. It is proved that in these circumstances, the result also holds true when $n = N + 1$. This slight variant is sometimes called the 'strong form' or the 'complete' version of the method. This means that the usual method is sometimes referred as the "weak form".

The usual illustration of the "strong" variant is the proof that every natural number greater that 1 is a unique product of primes.[9]

This is obviously true for small natural numbers: 2 is itself prime, as is 3, $4 = 2 \times 2$, 5 is prime, $6 = 2 \times 3$, etc. Consider some number N and suppose that all numbers up to and including N exhibit prime factorization. Now consider $N + 1$. Then either $N + 1$ is prime, in which case, we are done, or else it is not. If this is the case, then $N + 1$ will have factors, all of them less than N. But now, by the inductive hypothesis, these factors are themselves products of primes, and so their product $(N + 1)$ will also be a product of primes. End of story!

All the same, this is a case not unlike those discussed in my previous column. The factorization of numbers was one of the topics studied in my own primary education. It was only when I entered university, that I discovered that it was seen as requiring proof.

[9] Remember that 1 is not counted as a prime. I discussed this matter in my column of Vol. **41**, No. **2**.

I will continue the discussion of mathematical induction in my next column. There is a lot more to the story!