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A Bush Pilot's Tour

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In the Australian outback there are 20 cattle ranches which are visited every month by a veterinarian. On one of these aerial trips the veterinarian took along his assistant as a copilot. When they were flying between two ranches the assistant suddenly said:

"We just crossed the route we had been on a few hours ago. There must be a shorter tour than the one we are currently taking."

The vet smiled and asked:

"My friend, how do you know?

1. A mathematician I know told me that there are more than 2.4 quantillions (2.4×10^{18}) of different tours[1] between our airport and those 20 ranches.

2. You cannot possibly know which tour we will actually take.

3. You have never been on any such tour."

The assistant replied:

"Let's fly home. Then I shall demonstrate why I'm right."

The Assistant's Proof

To begin with, here is a mathematical model for the situation: In a two-dimensional plane (the outback), n points are given representing; A (point of take-off), and $B, C, \ldots$ (the ranches). The points are joined pairwise by straight lines (the possible routes between two stops). A system of connected lines is called a route r, if r begins in A and contains additional points $\in \{B, C, \ldots\}$. A route that begins and ends in A is a tour, if it contains each of the points $B, C, \ldots$.

Example:

Proposition 1

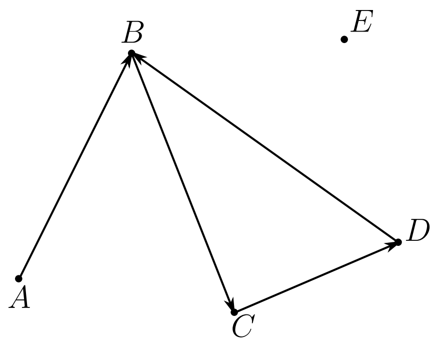
If a route r crosses itself, then there is a route r^* shorter than r.

Let $r = A \ldots BC \ldots DE \ldots F$ be a route, whose lines BC and DE intersect at the point X.

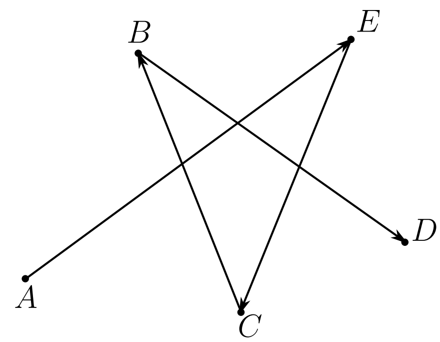
1. Suppose $X \neq A, B, C, D, E$. In r we replace BC by BD and DE by CE. The new route is $r^* = A \ldots BD \ldots CE \ldots F$. For r and r^* it holds that:
   Proposition 2

   The route r^* is shorter than r.

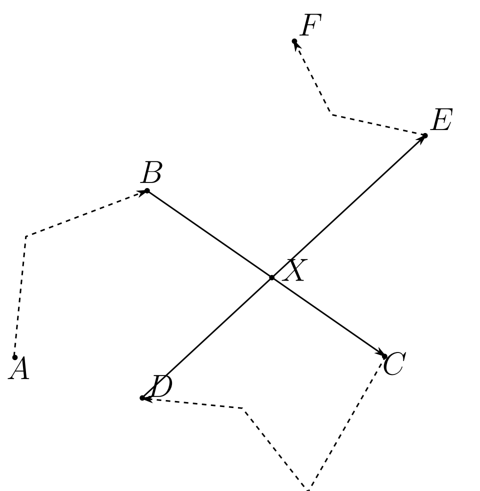
[1]Starting a tour in A there are 20 choices for the first ranch, 19 for the seoncd, 18 for the third and so on, until there is no choice at all for the 20th ranch. Therefore there are $20 \times 19 \times 18 \times \cdots \times 2\times$ different tours.



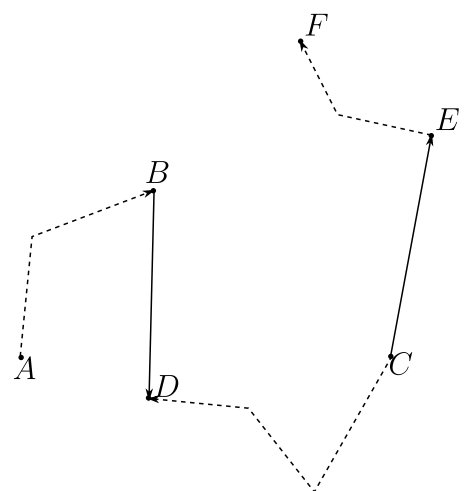
(a) The route $ABCDB$



(b) The route $ABCDE$



(c) The route $r = A \cdots BC \cdots DE \cdots F$



(d) The route $r^* = A \cdots BD \cdots CE \cdots F$

Proposition 2 results from the triangle inequality:

$$\begin{aligned} |BC| + |DE| &= |BX| + |CX| + |DX| + |XE| \\ &= (|BX| + |DX|) + (|CX| + |XE|) \\ &> |BD| + |CE|. \end{aligned}$$

2. Suppose now, that $X = B, C, D$ or E – e.g. $X = B$. If we replace BC in r with CE and omit BE, the new route is $r^* = A \ldots BD \ldots CE \ldots F$.

   The other cases are treated similarly and are thus left to the reader.

Again Proposition 2 holds because

$$|BC| + |DE| = |BC| + |DB| + |BE| = |BD| + (|BC| + |BE|) > |BD| + |CE|.$$

With Proposition 2 it follows: A bush pilot's tour which doesn't cross itself is shorter than a tour which does cross itself. Therefore the assistant's statement is true.

By the way: To find the shortest tour out of 2.4×10^{18} tours has an enormously higher degree of (numerical) complexity than the assistant's problem.