**Parabola Volume 49, Issue 3 (2013)**

# Solutions 1421–1430

**Q1421** Find all integer solutions of the equation $2013x + 49y = 2$.

**SOLUTION** Following the procedure set out in the article from the previous issue, we apply the Euclidean algorithm to $2013$ and $49$. Thus

$$\begin{aligned} 2013 &= 41 \times 49 + 4 \\ 49 &= 12 \times 4 + 1 \,. \end{aligned}$$

The final $1$ shows that $2013$ and $49$ have no common factor. Working backwards,

$$\begin{aligned} 1 &= 49 - 12 \times 4 \\ &= 49 - 12\,(2013 - 41 \times 49) \\ &= -12 \times 2013 + 493 \times 49 \,, \end{aligned}$$

and multiplying both sides (carefully!) by $2$ yields

$$2 = -24 \times 2013 + 986 \times 49 \,.$$

Comparing this with the desired equation shows that one possible solution is $x = -24$, $y = 986$, and the result from the article shows (remembering that we already know $a$ and $b$ have no common factor) that the complete solution is

$$x = -24 - 49t \,, \quad y = 986 + 2013t \,,$$

where $t$ is an integer.

**Q1422** Seven pirates are marooned on a deserted island. Before going to sleep they collect a pile of coconuts for food. During the night one of the pirates wakes up and decides to take his share. He divides the pile into seven equal parts and finds that there is one coconut left over, which he eats. He then puts six of the seven parts back into one pile and runs away with his share. Later two more pirates wake up. Not realising that one pirate has left, they also divide the pile into seven equal parts. There are two coconuts over; they eat these, put five shares back into one pile and run away with their own two shares. Later again three pirates wake up, divide the pile into seven, eat three leftover coconuts, take three of the seven shares and leave. The last pirate gets up in the morning. He sees that the other pirates have all gone, but he assumes they are out looking for more coconuts so he decides to share out those at the campsite. He finds that they divide into seven equal piles with none left over.

What is the smallest possible number of coconuts in the original pile?

**SOLUTION** Let $x$ be the number of coconuts in the original pile, $a, b, c$ the number in each share after the first three divisions, and $y$ the number in each share after the final division. Then

$$x = 7a + 1\ , \quad 6a = 7b + 2\ , \quad 5b = 7c + 3\ , \quad 4c = 7y\ .$$

Eliminating $a, b$ and $c$ from these equations leaves

$$120x - 2401y = 988\ . \tag{$*$}$$

We want $x$ and $y$ to be integers and so we use the standard method of solving linear Diophantine equations – see the article in the previous issue. Applying the Euclidean algorithm to $2401$ and $120$ is easy as it only has a single step! –

$$2401 = 20 \times 120 + 1\ .$$

So we have

$$120 \times (-20) - 2401 \times (-1) = 1\ ,$$

hence

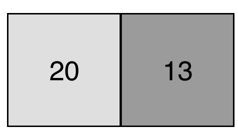
$$120 \times (-19760) - 2401 \times (-988) = 988$$

and by comparing with $(*)$ we can write down the general solution

$$x = -19760 + 2401t \quad \text{where } t \text{ is an integer.}$$

(We can also find $y$, but it is not needed for this problem). The smallest possible positive value for $x$ is obtained by taking $t = 9$, giving a pile of $x = 1849$ coconuts.

**Q1423** A rectangular dartboard has just two sections, which score $20$ and $13$ points respectively. With an unlimited number of darts available, is there a highest score not achievable? For instance, you can score $13, 20, 26, 33, 39, 40, 46$, but not any other score under $50$.




**SOLUTION** We need to determine what numbers can and cannot be expressed in the form $20x + 13y$, where $x, y$ are integers and $x \geq 0$, $y \geq 0$. Suppose that $a, b$ are positive integers with no common factor. We shall prove the following:

- if $c = ab - a - b$, then the equation $ax + by = c$ **does not** have a solution with $x, y \geq 0$;
- if $c > ab - a - b$, then the equation $ax + by = c$ **does** have a solution with $x, y \geq 0$.

First, suppose that

$$ax + by = ab - a - b\ . \qquad (*)$$

(We wish to show that this is not possible when $x, y$ are non–negative integers.) The equation can be written as

$$a(x+1) = b(a-y-1)\ .$$

This means that $b$ is a factor of $a(x+1)$; since $a$ and $b$ have no common factor, $b$ must be a factor of $x+1$. By a similar argument, writing $(*)$ as

$$b(y+1) = a(b-x-1)\ ,$$

we see that $a$ is a factor of $y+1$. Since $x+1$ and $y+1$ are positive integers, we have $x+1 \geq b$ and $y+1 \geq a$. If we rewrite $(*)$ once more as

$$a(x+1) + b(y+1) = ab\ ,$$

the left hand side is at least $2ab$ and the right hand side is $ab$, which is impossible. This proves the first statement.

Now let $c > ab - a - b$ and consider the equation $ax + by = c$. Since $a$ and $b$ have no common factor, we know from the article in the previous issue that the equation has solutions

$$x = x_0 - bt\ , \quad y = y_0 + at$$

for any integer $t$; the question remains whether or not we can find a value of $t$ such that $x, y \geq 0$. In fact, let $t$ be the largest integer less than $(x_0+1)/b$. Then we have

$$\frac{x_0+1}{b} - 1 \leq t < \frac{x_0+1}{b}$$

and so

$$x = x_0 - bt > x_0 - b\,\frac{x_0+1}{b} = -1\ ;$$

moreover,

$$y = y_0 + at \geq y_0 + a\Big(\frac{x_0+1}{b} - 1\Big) = \frac{ax_0 + by_0 + a - ab}{b}\ ,$$

and the facts that $ax_0 + by_0 = c$ and $c > ab - a - b$ give

$$y \geq \frac{c + a - ab}{b} > -\frac{b}{b} = -1\ .$$

So $x, y > -1$, but since $x, y$ are integers, this means that $x, y \geq 0$, and we have found a solution in non–negative integers.

In the case $a = 20$, $b = 13$ we have $ab - a - b = 227$, and this is the largest number not achievable on our dartboard.

**Q1424** In problem 1413 (see the solution last issue), suppose that we do not use all twenty–four numbers but remove those which are less than $1230$. What is now the smallest possible value of

$$|x-1230|+\cdots+|x-2013|+\cdots+|x-3210|\ ?$$

**SOLUTION** The numbers remaining are $1230$, $1302$, $1320$, those beginning with $2$ (six of these) and those beginning with $3$ (six more). So we now have fifteen numbers and we use the "odd case" of the solution to problem 1411. We have $n=15$ and $k=8$, so the minimum value is obtained when $x$ is the eighth smallest of the available numbers, which is $2301$. From formula $(*)$ in the solution of problem 1411, the minimum value is

$$-1230-\cdots-2130+2310+\cdots+3210$$

(note that $2301$ itself is missing – it is neither added nor subtracted). The total of all these numbers is $8847$.

**Q1425** As in problem 1414, an $(a,b)$–*superknight* is a knight–like chess piece which moves $a$ squares up, down, left or right and then $b$ squares in a perpendicular direction. In problem 1408 we saw that a $(3,8)$–superknight can move from a square to the square immediately to its right in thirteen moves: we have

$$5(8,-3)+3(-8,-3)+(-3,-8)+4(-3,8)=(1,0)\ ,$$

and the number of moves on the left hand side is $5+3+1+4=13$. Next question: is it possible to do this in fewer than $13$ moves?

**SOLUTION** No, it is not. To move the $(3,8)$–superknight one square to its right we need

$$w(8,-3)+x(-8,-3)+y(-3,-8)+z(-3,8)=(1,0)$$

where $w,x,y,z$ are integers, and the number of moves taken will be $|w|+|x|+|y|+|z|$. (Note that we can't "subtract" moves – a coefficient of $-5$ would still count as $5$ moves.) Separating out the right/left moves and the up/down moves, we have

$$8w-8x-3y-3z=1 \quad \text{and} \quad -3w-3x-8y+8z=0\ .$$

Adding and subtracting these equations yields

$$5(w+z)-11(x+y)=1 \quad \text{and} \quad 11(w-z)-5(x-y)=1\ .$$

We solve the first equation by regarding $w+z$ as a single variable, $x+y$ likewise, and treating it as a linear Diophantine equation; similarly for the second; this gives

$$w+z=-2+11s\ , \qquad x+y=-1+5s$$
$$w-z=1+5t\ , \qquad x-y=2+11t$$

for some integers $s, t$; finally, we derive formulae for $w, x, y, z$ by adding and subtracting these,

$$2w = -1 + 11s + 5t \;, \qquad 2x = 1 + 5s + 11t \;,$$
$$2y = -3 + 5s - 11t \;, \qquad 2z = -3 + 11s - 5t \;.$$

Now if $|w| + |x| + |y| + |z| < 13$ then $|2w| \le 24$ and so on; hence

$$-24 \le -1 + 11s + 5t \le 24$$
$$-24 \le 1 + 5s + 11t \le 24$$
$$-24 \le -3 + 5s - 11t \le 24$$
$$-24 \le -3 + 11s - 5t \le 24 \;.$$

Adding the first and fourth inequalities, subtracting the second and third, gives

$$-48 \le -4 + 22s \le 48 \quad \text{and} \quad -48 \le 4 + 22t \le 48$$

and so $s = -2, -1, 0, 1, 2$ and $t = -2, -1, 0, 1, 2$. From previous equations $s + t$ is odd, which leaves twelve pairs to check; the values of $|w| + |x| + |y| + |z|$ calculated in the following table

| | $s = -2$ | $s = -1$ | $s = 0$ | $s = 1$ | $s = 2$ |
|---|---|---|---|---|---|
| $t = -2$ | | 33 | | 29 | |
| $t = -1$ | 35 | | 13 | | 29 |
| $t = 0$ | | 19 | | 13 | |
| $t = 1$ | 37 | | 19 | | 33 |
| $t = 2$ | | 37 | | 35 | |

show that the number of moves is never less than 13.

**Comment** . . . and one of the 13–move solutions has $s = 1$, $t = 0$, giving $w = 5$, $x = 3$, $y = 1$, $z = 4$, the solution we have found already.

**Q1426** The number 132 is a multiple of 1, and of 3, and of 2. What is the largest number you can find which has all its digits different, does not contain zero, and is a multiple of each of its digits?

**SOLUTION** First assume that the digit 5 is not used. If all remaining non–zero digits are used then the sum of the digits is $1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 = 40$; as 40 is not a multiple of 9 this is impossible, and we can therefore only use (at most) seven digits. For this to work, the digits must include at least one of 3 and 9; therefore the sum of the digits is a multiple of 3; therefore the missing digit is 1, 4 or 7; therefore 9 is not the missing digit, so the sum is actually a multiple of 9, and the missing digit has to be 4.

So, we are attempting to find the largest number (if any – conceivably we will have to go down to six digits) consisting of the digits $1, 2, 3, 6, 7, 8, 9$ which is a multiple of all these digits. Any number consisting of these digits will certainly be a multiple of $1, 3$ and $9$, so we need not consider these any further; and if it is a multiple of $8$ then it will automatically be a multiple of $2$ and of $6$. Therefore we are looking for a number which is a multiple of $7$ and $8$.

To find the largest possible number we try first something beginning $9876\cdots$. If it is to be a multiple of $8$ then the last digit has to be even, so there are two possibilities, $9876312$ and $9876132$. However the first of these is not a multiple of 7, and the second is not a multiple of $8$. The next largest possibility is $9867312$, and it is not difficult to check that this works.

It remains to consider the possibility of including $5$ as one of the digits. In this case we cannot have any even digits, as this would give a number which is a multiple both of $2$ and of $5$: such a number is a multiple of $10$ and must end in $0$, which is forbidden. So the number could only contain at most five digits, and cannot be bigger than $9867312$. Therefore, the largest possible answer is $9867312$.

**Q1427** As in problem 1419, we wish to write a given number $n$ as a sum of positive integers in such a way that the product of the summands is as large as possible. For this question, however, the summands must be $4$ or more. For example, $14 = 4 + 5 + 5$ is allowed but $14 = 2 + 3 + 4 + 5$ is not. If $n = 2013$, what is the maximum product we can obtain in this way?

**SOLUTION** The equations

$$8 = 4 + 4\ , \quad 9 = 4 + 5\ , \quad 10 = 4 + 6$$

and so on show that we can never obtain the largest possible product if we use summands of $8$ or more, because in each case the product of summands on the right hand side exceeds the left hand side. For the same reason, the equations

$$7 + 7 = 4 + 4 + 6\ , \quad 7 + 6 = 4 + 4 + 5\ , \quad 7 + 5 = 4 + 4 + 4\ , \quad 7 + 4 = 5 + 6$$

show that we can never use a $7$. Thirdly,

$$6 + 6 = 4 + 4 + 4\ , \quad 6 + 5 + 5 = 4 + 4 + 4 + 4\ , \quad 6 + 4 = 5 + 5$$

show that we cannot have two 6s; if we have a $6$ then we cannot have two $5$s; and if we have a $6$ then we cannot have a $4$. Finally,

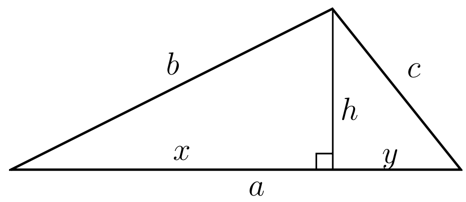
$$5 + 5 + 5 + 5 = 4 + 4 + 4 + 4 + 4$$

shows that we cannot have more than three $5$s. So, how can we find suitable numbers adding up to $2013$? If we don't use any $4$s then the most we can get is $6 + 5$ or $5 + 5 + 5$, which are both way too small. If we do have $4$s then we cannot have $6$s and so we must make up $2013$ from $4$s and $5$s only, with at most three $5$s. The only possibility is

$$2013 = 4 + \cdots + 4 + 5$$

with one $5$ and all the rest $4$s. Therefore the maximum possible product is $5 \times 4^{502}$.

**Q1428** Use the following diagram to find the area of a triangle in terms of its side lengths $a, b$ and $c$. (The answer is a formula which you may know already.)




**SOLUTION** By Pythagoras' Theorem we have

$$x^2 + h^2 = b^2 \quad \text{and} \quad y^2 + h^2 = c^2 \ ;$$

from the diagram,

$$x + y = a \ . \tag{1}$$

By subtracting the first two equations we have $x^2 - y^2 = b^2 - c^2$, and substituting these results into the identity

$$(x + y)^2(x - y) = (x^2 - y^2)(x + y)$$

yields

$$a^2(x - y) = (b^2 - c^2)(x + y) \ .$$

Collecting all the $x$s on the left hand side and all the $y$s on the right,

$$(a^2 - b^2 + c^2)x = (a^2 + b^2 - c^2)y \ . \tag{2}$$

Now multiply both sides of $(1)$ by $a^2 + b^2 - c^2$ and substitute from $(2)$; we obtain

$$(a^2 + b^2 - c^2)x + (a^2 - b^2 + c^2)x = (a^2 + b^2 - c^2)a \ ,$$

and simplifying yields

$$2ax = a^2 + b^2 - c^2 \ .$$

If $A$ is the area of the triangle then $A = \frac{1}{2}ah$ and therefore

$$16A^2 = 4a^2h^2 = 4a^2(b^2 - x^2) = (2ab + 2ax)(2ab - 2ax) \ ,$$

that is,

$$16A^2 = (2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2) \ .$$

We can simplify the right hand side by using the difference of two squares a number of times,

$$\begin{aligned} 16A^2 &= ((a + b)^2 - c^2)(c^2 - (a - b)^2) \\ &= (a + b + c)(a + b - c)(c + a - b)(c - a + b) \ . \end{aligned}$$

It is customary to write this in terms of the semi–perimeter $s$ of the triangle, given by $2s = a + b + c$, to obtain

$$16A^2 = (2s)(2s - 2c)(2s - 2b)(2s - 2a)$$

and hence, finally,

$$A = \sqrt{s(s-a)(s-b)(s-c)}\ .$$

This is known as **Heron's formula**.

**Q1429** For what values of the coefficients $a, b, c, d$ can the quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

be solved by using the method of problem 1412 (solution in the previous issue)?

**SOLUTION** We need to find coefficients $k$ and $l$ such that

$$x^4 + ax^3 + bx^2 + cx + d = (x^2 + kx)^2 + l(x^2 + kx) + d\ . \qquad (*)$$

Expanding the right hand side and equating coefficients of $x^3$, $x^2$ and $x$, we have

$$a = 2k\ , \quad b = k^2 + l\ , \quad c = kl\ .$$

Solving the first two equations for $k$ and $l$ in terms of $a$ and $b$, then substituting into the third, gives

$$8c = a(4b - a^2)\ ;$$

for the method to succeed it is necessary that this condition hold. Conversely, if that condition is true and we choose $k = \frac{1}{2}a$, $l = b - \frac{1}{4}a^2$, then it is easy to check that $(*)$ is true, and we can proceed to solve the quartic as in the earlier problem.

**Comment**. We could also try looking for something like

$$x^4 + ax^3 + bx^2 + cx + d = (x^2 + kx + m)^2 + l(x^2 + kx + m) + d\ ,$$

but if you think about it you should be able to see that this will not give us any further possibilities.

**Q1430** Find all solutions of the equation

$$x^2 - 12\lfloor x \rfloor + 23 = 0\ ,$$

where $\lfloor x \rfloor$ denotes the integer part of $x$, that is, $x$ rounded *down* to the nearest integer.

**SOLUTION** Let $n = \lfloor x \rfloor$. Then $x^2 = 12n - 23$, so that $n \geq 2$; and hence

$$x = \sqrt{12n - 23}\ ,$$

taking the positive root since $x \geq n > 0$. Since $n \leq x < n + 1$ we have

$$n \leq \sqrt{12n - 23} < n + 1\ ;$$

this gives $n^2 - 12n + 23 \leq 0$ and $n^2 - 10n + 24 > 0$. Calculation shows that the first quadratic has a root between $2$ and $3$ and another between $9$ and $10$; since $n$ is an integer and the quadratic has a negative value, we have

$$3 \leq n \leq 9\,.$$

Similarly the second inequality shows that

$$\text{either} \quad n \leq 3 \quad \text{or} \quad n \geq 7\,.$$

Combining all the information we have found gives four possibilities $n = 3, 7, 8, 9$, and hence the four solutions

$$x = \sqrt{13}\,,\ \sqrt{61}\,,\ \sqrt{73}\,,\ \sqrt{85}\,.$$