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# Problems 1441–1450

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**Q1441** Hundreds of millions of pebbles were deposited on the foreshore of a beach. Every high tide, 20% of the pebbles are transported from the deep end to shallow end. Every low tide, 14% of the pebbles are transported from the shallow to the deep end. Over a long period of time, what is the exact fraction of the pebbles on the deep end after high tide?

**Q1442** $ABCD$ is a rectangle consisting of $2014 \times 1729$ squares. A particle is dispatched from $A$ at $45^\circ$ and bounces off $BC$ then $CD$, and so on. If it reaches one of the corners $A, B, C, D$ it falls through a tiny hole and leaves the rectangle. Will this ever happen? If so, which corner will the particle fall through, and how many walls will it hit during its journey?

**Q1443**

**(a)** Factorise the quartic $x^4 + 6x^2 - 16x + 9$.

**(b)** Use the previous result to find the maximum value of $\dfrac{x}{(x^2+3)^2}$.

**(c)** Find the maximum area of a triangle inscribed within the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as shown: one vertex of the triangle is to be at the point $(0, -b)$ and the opposite side is to be horizontal.

**Q1444** In a certain town each of the inhabitants is either a truth–teller or a liar; however this does not mean that everyone is actually able to answer every question they are asked. If a truth–teller is absolutely certain of the answer to a question, he will give that answer; if not, he will say, "I don't know". On the other hand, a liar will never truthfully admit to not knowing something: he will give an answer that he knows is false, if any, but if there is nothing that he is *certain* is false then he will give a randomly chosen answer (possibly even, by accident, the true answer). Moreover, everyone in this town can instantly deduce the logical consequences of any facts they know.

I meet four inhabitants of this town and ask them, "How many of you four are truth–tellers?"

Kevin says, "I don't know"; then Laura says, "One"; then Mike says, "None". Noela, however, is asleep. Fortunately I don't need to wake her up, since I can already tell whether she is a truth–teller or a liar. Which is she?

**Q1445** The *Lucas sequence*, a famous sequence of numbers closely related to the even more famous Fibonacci sequence, is

$$1,\ 3,\ 4,\ 7,\ 11,\ 18, \ldots\ :$$

where every number in the sequence, apart from the first two, is the sum of the previous two numbers. Prove that if we take every second number in the Lucas sequence, that is, $3, 7, 18, \ldots$ and so on for ever, we will never find a number which is a perfect square.

**Q1446** Use the Peano axioms for the natural numbers (see Michael Deakin's article in this issue) to prove the cancellation law for addition: for all numbers $a, b, c$, if $a + c = b + c$ then $a = b$. The basic facts you may use are the five axioms, Michael's equations $(1)$ and $(2)$ which define addition, and *nothing else*. In particular, you cannot just say "subtract $c$ from both sides", because subtraction has not yet been defined!

**Q1447** If the roots of the equation $x^3 - x + 2014 = 0$ are $\tan\alpha,\ \tan\beta$ and $\tan\gamma$, evaluate $\tan(\alpha + \beta + \gamma)$.

**Q1448** Let $f(x)$ be a polynomial with degree $2012$, such that

$$f(1) = 1\ , \quad f(2) = \frac{1}{2}\ , \quad f(3) = \frac{1}{3}\ , \ldots , \quad f(2013) = \frac{1}{2013}\ .$$

Find the value of $f(0)$.

**Q1449**

(a) A triangle has angles $\alpha$ and $\beta$, the side between these two angles being of length $c$. Show that the area of the triangle is given by

$$A = \frac{1}{2}\, c^2\, \frac{\sin\alpha \sin\beta}{\sin(\alpha + \beta)}\ .$$

(b) The square $ABCD$ has side length $1$. A line is drawn through $A$ intersecting $BC$; it makes an angle $\alpha$ with the diagonal through $A$. A line is drawn through $B$ intersecting $CD$; it makes an angle $\beta$ with the diagonal through $B$. Lines are drawn through $C$ and $D$ parallel to those through $A$ and $B$ respectively. If the parallelogram determined by these four lines is removed from the square, find the remaining area.

**Q1450** Consider again the coin–sharing question in problem 1435 (stated and solved later this issue). Suppose that before the five people have the chance to share out the $10$ coins they are joined by Fiona, George and Harry. The coins are then shared out by means of a similar procedure, with Harry having the first opportunity to propose a distribution. One extra condition needs to be stated: if a person making a proposal for the distribution of the coins has two or more options which are equally good for him, then he will choose between them at random, so for example, by writing them down on slips of paper and pulling one out of a hat.

How many coins will each person receive in this situation?