Parabola Volume 50, Issue 1 (2014)

Solutions 1431–1440

Q1431 Find a four–digit number with the following property: if the last digit of the number is moved to the front and 7 is added to the result, the answer is exactly twice the original number. Is there more than one such number?

SOLUTION Let x consist of the first three digits of the number and let y be the last digit. Then the number is $10x + y$, and the stated condition says that

$$1000y + x + 7 = 2(10x + y) .$$

This simplifies to

$$19x - 998y = 7 , \qquad (*)$$

a linear Diophantine equation which we can solve by the method explained in volume 49, number 2, and used in some of last issue's solutions. Applying the Euclidean algorithm to 998 and 19, we have

$$\begin{aligned} 998 &= 52 \times 19 + 10 \\ 19 &= 1 \times 10 + 9 \\ 10 &= 1 \times 9 + 1 . \end{aligned}$$

The final 1 shows that 998 and 19 have no common factor, and so the equation $(*)$ has integer solutions. Working backwards,

$$\begin{aligned} 1 &= 10 - 9 \\ &= 10 - (19 - 10) \\ &= -19 + 2 \times 10 \\ &= -19 + 2\,(998 - 52 \times 19) \\ &= 2 \times 998 - 105 \times 49 , \end{aligned}$$

and multiplying both sides by 7 yields

$$7 = 14 \times 998 - 735 \times 19 .$$

Comparing this with the desired equation shows that the solution is

$$x = -735 + 998t , \quad y = -14 + 19t ,$$

where t is an integer. As y is a single digit, the only possibility is $t = 1$, which gives $y = 5$ and $x = 263$. Therefore the number sought is 2635.

Q1432 Use the method of mathematical induction (if you have not yet learned it at school, it is explained in Michael Deakin's article in the previous issue) to prove that every one of the numbers

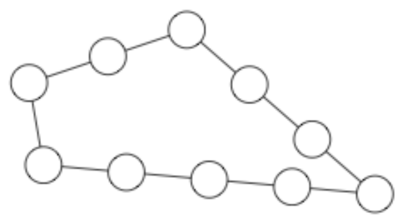
$$171\,,\ 17271\,,\ 1727271\,,\ 172727271\,,\ 17272727271\,,\ldots$$

is a multiple of 19.

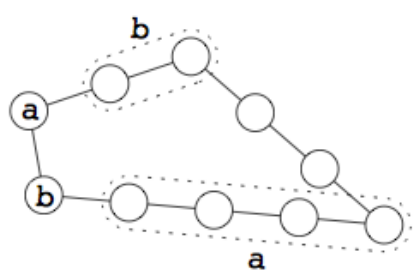
SOLUTION As $171 = 9 \times 19$, the first number is a multiple of 19. Each number in the list (except the first) is 100 times the previous number, plus 171. Therefore, if any particular number in the list is a multiple of 19, so is the following one. By mathematical induction, every number in the list is a multiple of 19.

Comment. This problem was set in *Parabola incorporating Function* volume 47, number 3, and was solved in the following issue *without* using mathematical induction.

Q1433 Fill in the circles in the diagram with the digits 0 to 9, in such a way that no two consecutive digits occur in adjacent circles, and the sum of the numbers on each side of the quadrilateral is the same.



SOLUTION As shown in the diagram below, we denote the two numbers at the left by a and b. Since the sum of the digits on each side is $a + b$, the two digits indicated at the top of the diagram must add up to b, and the four at the bottom must add up to a.



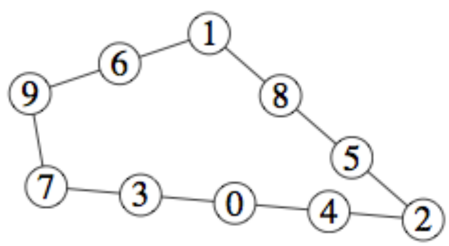
As the total of all ten digits is 45, the two unmarked digits on the right must add up to $45 - 2a - 2b$. The total of all four digits on the right is at least $45 - 2a - 2b + 0 + 1$; but this total must in fact equal $a + b$, the same as that of the other three sides. Therefore

$$a + b \geq 46 - 2a - 2b\ ,$$

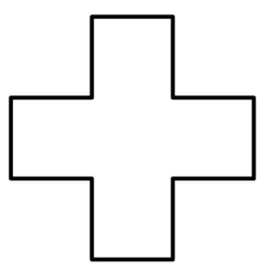
which shows that $3a+3b \geq 46$ and hence $a+b \geq 16$. Since a and b are two different and non–consecutive digits, they must be 7 and 9. The sum of the two digits in the middle of the right-hand side is $45 - 2a - 2b = 13$. These digits cannot be 9 and 4 as 9 has already been used; nor can they be 7 and 6; they must be 8 and 5. The two numbers within the dotted line at the top of the diagram must add up to 7 or 9: what are they?

- They cannot be 4 and 3 as there would then be consecutive digits in adjacent circles.
- They cannot be 6 and 3: if the 6 were the right-hand digit then the sum of the four numbers on the right would be too big, while if it were the left-hand digit it would be next to a 7.
- The only possibility remaining is that these numbers are 6 and 1.

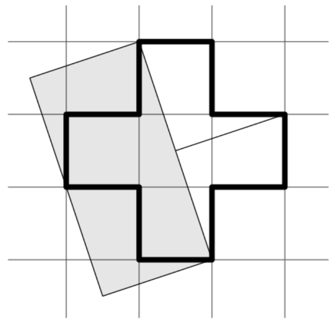
Therefore the four numbers inside the lower dotted curve are $0, 2, 3, 4$ and it is easy to place them. One solution is shown below; the only other possibility is to interchange the 5 and 8.



Q1434 In the figure shown below all sides have equal length and each angle is a right angle. Show how to cut the figure into three pieces which can be rearranged to form a rectangle with sides in the ratio $2:1$.



SOLUTION The given figure is outlined in thick strokes in the following diagram, and a grid has been added to assist the explanation. The length of the side does not matter; for convenience we suppose it is 2. Then the figure has area 20, and a 2 by 1 rectangle with this area has sides $\sqrt{10}$ and $\sqrt{40}$. A suitable rectangle is shaded; it is easy to see from Pythagoras' Theorem that it has the correct dimensions, because $10 = 3^2 + 1^2$ and $40 = 6^2 + 2^2$.



The shaded rectangle is divided into three pieces; one of these overlaps half of the cross, and the other two can be formed by cutting the remaining half into two pieces.

Q1435 Adam, Betty, Cathy, Doug and Ellen have 10 coins, all of equal value, and are going to share them by the following procedure.

- Ellen will propose a distribution of the coins (so many to herself, so many to Doug and so on). The other four will vote on it. If 50% or more votes are in favour of Ellen's proposal it is accepted and the coins are distributed accordingly.
- If Ellen's proposal is rejected then she receives no coins and goes home. In this case Doug makes a proposal which is voted upon by the other three. Once again, if 50% or more are in favour then Doug's proposal is accepted; if not, Doug goes home with nothing.
- As long as proposals are rejected, the procedure continues, with one person fewer each time. If, eventually, Betty has a proposal rejected, then Adam gets all the coins.

How many coins will each person receive? We assume that all the participants are rational, and that they know the others are rational too. "Rational" means that they are able to work out the precise consequences of their actions; and that a person proposing a distribution of the coins will make the proposal which will bring him/her the maximum possible number of coins; and that each person will vote against a proposal if doing so *cannot* give him a worse outcome than voting for it, but will vote for it otherwise.

SOLUTION Adam thinks: if Betty makes any proposal at all I can vote against it and get all 10 coins for myself.

Betty thinks: I know what Adam is thinking. If I have to make a proposal he will vote against it, I will lose and will get nothing at all. So if Cathy makes a proposal which offers me one coin or more I will vote for it, otherwise I will vote against.

Cathy thinks: I know what Betty is thinking. She will definitely support me if I offer her one coin or more; on the other hand, Adam will never support me, so there is no point in my offering him anything at all. So if Doug's proposal is defeated, my best course is to propose 9 coins for myself, 1 for Betty and 0 for Adam. Betty will vote for this (because if it is defeated she gets even fewer coins) and so it will be accepted.

Doug thinks: I know what Cathy is thinking. She is planning to offer Betty 1 coin and Adam nothing, and they know it too. So to get two votes and have my proposal accepted, I need to get them onside by offering them more. Cathy will vote against me (and in any case I don't need her vote) so I won't offer her anything. I will propose nothing for Cathy, 2 coins for Betty and 1 for Adam, which leaves 7 for myself. If I make this proposal it will certainly be accepted, because Betty and Adam will vote for it.

Ellen thinks: I know what Doug is thinking. I need two of the four votes to get my proposal accepted. The "cheapest" way for me to do this is to offer 1 coin to Cathy and 2 to Adam: they will then vote for me because they know that they will do even worse if my proposal is defeated.

So, the outcome is that Ellen will receive 7 coins, Doug 0, Cathy 1, Betty 0, and Adam 2. This distribution will be proposed by Ellen; Cathy and Adam will vote for it and so it will be accepted. In the event, nobody other than Ellen ends up actually making the proposal they were intending.

Q1436 We saw in problem 1430 (solution in the previous issue) that the equation $x^2 - 12\lfloor x \rfloor + 23 = 0$ has four solutions, where the notation $\lfloor x \rfloor$ denotes x rounded to the nearest integer downwards. Show how to find positive integers a and b for which the equation

$$x^2 - 2a\lfloor x \rfloor + b = 0$$

has as many solutions as desired. In particular, find an equation of this type which has more than 2013 solutions.

SOLUTION As in the previous solution, we write $n = \lfloor x \rfloor$, so that $x = \sqrt{2an - b}$. This will give a solution of the equation provided that

$$n \le \sqrt{2an - b} < n + 1 \;;$$

a bit of algebra leads to

$$n^2 - 2an + b \le 0 \quad \text{and} \quad n^2 - (2a - 2)n + b + 1 > 0 \,.$$

Therefore a solution will exist for every value of n which is between the larger root of the second quadratic and the smaller root of the first:

$$a - 1 + \sqrt{a^2 - 2a - b} < n < a + \sqrt{a^2 - b} \,.$$

Now there will certainly be k or more integers n which satisfy this inequality if the difference between the left-hand side and right-hand side is more than k. We have

$$\text{RHS} - \text{LHS} = 1 + \sqrt{a^2 - b} - \sqrt{a^2 - 2a - b} \,,$$

and we can make this large by choosing the smallest possible value for b. Therefore we take $b = a^2 - 2a$, which gives

$$\text{RHS} - \text{LHS} = 1 + \sqrt{2a}.$$

This can be made as large as we like by suitably choosing a, and so we obtain as many solutions as desired. In particular, to get more than 2013 solutions we take $1+\sqrt{2a} > 2014$, that is, $a > 2013^2/2$. For example, we could take $a = 2026085$ and $b = a^2 - 2a = 4105016375055$. Thus, the equation

$$x^2 - 4052170\lfloor x\rfloor + 4105016375055 = 0$$

has more than 2013 solutions.

Q1437 As in previous questions, $\lfloor x\rfloor$ denotes x rounded to the nearest integer downwards. Prove that if we calculate the expressions n^2+n-1 and $n+\lfloor\sqrt{n}\rfloor$ for $n = 1, 2, 3, \ldots$, we obtain all the positive integers once each.

SOLUTION If we calculate the values of the two given expressions for $n = 1, 2, 3, \ldots$, we obtain

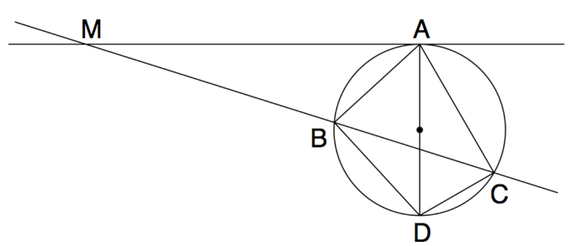
$$\begin{array}{rl} n^2+n-1 = & 1, \qquad\qquad 5, \qquad\qquad\qquad 11, \\ n+\lfloor\sqrt{n}\rfloor = & \quad 2,\ 3,\ 4, \quad 6,\ 7,\ 8,\ 9,\ 10, \quad 12, \ldots \end{array}$$

To prove that the pattern continues, consider the values of $n+\lfloor\sqrt{n}\rfloor$ for $n = a^2, a^2+1, a^2+2, \ldots, (a+1)^2-1$: that is, n goes from a square up to just short of the next square. For such n the value of $\lfloor\sqrt{n}\rfloor$ is always equal to a, so as n increases we obtain all the values of $n+a$ from a^2+a to $(a+1)^2+a-1$, with none missing and no repetitions.

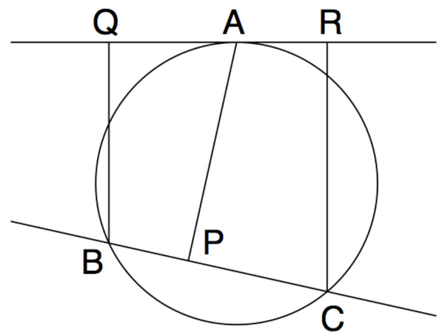
If $a \ge 2$ then the ath group of numbers begins with a^2+a, and the previous group ends with $((a-1)+1)^2+(a-1)-1$, that is, a^2+a-2. Therefore there is only one missing number between the two groups, and it is a^2+a-1. Therefore, if $n = 2, 3, 4, \ldots$ then the expression n^2+n-1 covers all these missing numbers, once each; while if $n = 1$ then n^2+n-1 is 1, which is the number missing before the first group. Therefore, the two expressions collectively give all the positive integers once each.

Q1438 A line is tangent to a circle at the point A; another line cuts the circle at two points B and C; these two lines meet at M. Prove that $(MB)(MC) = (MA)^2$.

SOLUTION The situation is shown in the diagram, with a diameter AD of the circle included. Since angles standing on the same chord of a circle are equal, we have $\angle BAD = \angle BCD$. Subtracting these from the right angles $\angle MAD$ (angle between the tangent and diameter) and $\angle ACD$ (angle in a semicircle) gives $\angle MAB = \angle MCA$. The triangles $\triangle MAB$ and $\triangle MCA$ are similar, because they have a common angle at M and equal angles $\angle MAB$, $\angle MCA$ as we have just shown. Therefore $MA/MB = MC/MA$, and the result follows.



Q1439 In the diagram, the line QR is tangent to the circle at A; the angles at P, Q and R are right angles. Prove that $(BQ)(CR) = (AP)^2$.



SOLUTION If the line BC is parallel to the tangent then $BQ = AP = CR$ and the result is clearly true. Otherwise, let M be the point of intersection of BC and the tangent. Then $\triangle MAP$ and $\triangle MBQ$ and $\triangle MCR$ are all similar, since they have a common angle and a right angle. Therefore $BQ/MB = AP/MA = CR/MC$ and so

$$\frac{BQ}{MB}\,\frac{CR}{MC} = \frac{(AP)^2}{(MA)^2}\ .$$

But by the result of the previous question, the denominators on the left-hand side and right-hand side are equal; therefore the numerators are also equal, and this is what we wanted to prove.

Q1440 Let $f(x)$ be a polynomial with degree 2012, such that

$$f(1) = 1\ , \quad f(2) = \frac{1}{2}\ , \quad f(3) = \frac{1}{3}\ , \ldots , \quad f(2013) = \frac{1}{2013}\ .$$

Find the value of $f(2014)$.

SOLUTION We can write the given equations as

$$xf(x) - 1 = 0$$

for $x = 1, 2, 3, \ldots, 2013$. Since $xf(x) - 1$ is a polynomial with degree 2013 which has 2013 roots, we can give its factorisation, involving one unknown constant:

$$xf(x) - 1 = c(x-1)(x-2)\cdots(x-2012)(x-2013)\ . \qquad (*)$$

By substituting $x = 0$ we obtain

$$-1 = c(-1)(-2)\cdots(-2012)(-2013)$$

and so $c = 1/2013!$. By substituting this value back into equation $(*)$ and taking $x = 2014$ we find

$$2014f(2014) - 1 = \frac{1}{2013!}(2013)(2012)\cdots(2)(1) = 1$$

and so

$$f(2014) = \frac{2}{2014} = \frac{1}{1007} \, .$$