Parabola Volume 50, Issue 2 (2014)

History of Mathematics: Making the Imaginary Real and Respectable

Michael A B Deakin[1]

Back in 2005, I devoted two of these columns to the history of complex and imaginary numbers. Here I return to the theme, but take a different slant on it, telling how an initially suspect notion became respectable. Let me begin by recapping the source of the difficulty. We have the basic rules of arithmetic that tell us that:

$$+ \times + = - \times - = +; \; + \times - = - \times + = -.$$

It seems that there is no possibility of finding $\sqrt{-1}$. Yet, as outlined in my earlier treatment, the notion proved useful (indeed, more than that, necessary) for a complete account of the cubic equation. When I was still in High School, my Maths teacher airily remarked: "There's not really a problem; you just invent a square root, $\sqrt{-1} = i$, and proceed as if nothing untoward has happened". In other words, if we do this, it works, and really no more needs to be said. Something like this attitude must have informed the mathematics practiced for some 400 years before his remark. The number i was designated as "imaginary", but dealt with exactly as if it were an ordinary, common or garden, real number.

Numbers that combined the familiar "real" numbers with these new "imaginary" ones were designated as "complex", but otherwise subjected to exactly the same rules of arithmetic as the ordinary "real" numbers. So, if $\alpha + i\beta$ were one complex number and $\gamma + i\delta$ another, these numbers could be added: $(\alpha + i\beta) + (\gamma + i\delta) = (\alpha + \gamma) + i(\beta + \delta)$, subtracted: $(\alpha + i\beta) - (\gamma + i\delta) = (\alpha - \gamma) + i(\beta - \delta)$ and multiplied: $(\alpha + i\beta) \times (\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\beta\gamma + \alpha\delta)$, where in this last computation, we have used the result $i^2 = -1$, but otherwise proceeded perfectly normally.

When it came to division, a subterfuge was (and still is) employed; one that probably seemed more natural when I was a boy than it does today, so I will digress for a while to provide background. Suppose you want to calculate $1/\sqrt{2}$, that is to say $1/1.4142...$. Well today this is just a matter of pressing a couple of buttons on your calculator, but back then this was not a possibility, and $1.4142...$ is a difficult number to divide by if you are doing it by hand. So one did a little preliminary work: $1/\sqrt{2} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \sqrt{2}/2 \approx 0.7071...$. Suddenly a difficult calculation has been made trivially easy.

[1]Dr Michael Deakin is an Adjunct Senior Research Fellow in the School of Mathematical Sciences at Monash University.

A rather more typical example is provided by the problem $1/\left(1+\sqrt{2}\right)$. Here the procedure is to write $\frac{1}{1+\sqrt{2}} = \frac{1-\sqrt{2}}{\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)} = \frac{1-\sqrt{2}}{1-2} = \sqrt{2}-1 \approx 0.4142...$. Notice the cunning use of the "difference of two squares" formula.

This procedure was referred to as "rationalizing the denominator". So, when it comes to complex numbers, we rationalize the denominator (or, more precisely, *realize* it). The general formula, whose derivation I leave to the reader, is

$$\frac{\alpha+i\beta}{\gamma+i\delta} = \frac{\alpha\gamma+\beta\delta}{\gamma^2+\delta^2} + i\frac{\beta\gamma-\alpha\delta}{\gamma^2+\delta^2}.$$

(But note that the "real" part, i.e. the first term, is a real number and the "imaginary"part, i.e. the second term, with its factor of i, is an imaginary number.)

So, we can simply follow the teacher's advice and "proceed as if nothing untoward has happened".

However, our difficulties as students arose because this wasn't at all how our textbook dealt with the matter. The text defined a complex number as a pair of real numbers $[\alpha, \beta]$ subject to a law of addition $[\alpha, \beta] + [\gamma, \delta] = [\alpha+\gamma, \beta+\delta]$ and a law of multiplication $[\alpha, \beta] \times [\gamma, \delta] = [\alpha\gamma - \beta\delta, \beta\gamma + \alpha\delta]$. It was pointed out that the subclass $[\alpha, 0]$ of the complex numbers had "properties much like those of the real" numbers, indeed were "sometimes called" real, while those of the form $[0, \beta]$ were referred to as "imaginary". In particular, it was customary to write $[0, 1] = i$. One can readily check that $[0, 1]^2 = [-1, 0]$, in other words $i^2 = -1$.

But the question that nagged with us was "Why bother with all this rigmarole, when the teacher's offhand remark dealt with the matter perfectly well?".

The textbook's approach was usually attributed to the Irish mathematician William Rowan Hamilton (1788-1856), and, of course, he didn't introduce it just to be perverse. What worried him (and in truth other mathematicians along with him) was the logical justification of the complex numbers. It was all very well to ignore the problem and "proceed as if nothing untoward has happened", but this procedure of ignoring the difficulty didn't make it go away!

What Hamilton did was to describe complex numbers in quite ordinary terms. There was nothing "imaginary" involved at all. We could sum it up by saying that the second ("imaginary") component was no more *imaginary* than the first and the first ("real") component no more *real* than the second.

Hamilton's approach also showed very clearly the relation of the complex numbers to the geometry of the plane, because the complex number $[\alpha, \beta]$ could be represented as the point $[\alpha, \beta]$ in the usual co-ordinate system. Indeed just as the real numbers correspond *exactly* to the points on a "number line", so the complex numbers correspond *exactly* to the points of a plane.

Mathematicians before Hamilton must have intuitively grasped this connection. Although Hamilton wrote in 1837, complex numbers had been in the public domain ever since the 1545 publication of Girolamo Cardano's *Ars Magna*. In the intervening years, they had been studied by (*inter alia*) Johann I Bernoulli (1667-1748), Abraham de Moivre (1667-1754), Roger Cotes (1682-1726) and Leonhard Euler (1707-1788). Many of the discoveries made in the course of these people's work made at least implicit reference to the geometric interpretation,

A clear case of the use of the geometric approach, however, occurs in the work of Carl Gauss (1777-1855). Gauss stated and proved what is now called *The Fundamental Theorem of Algebra*. Indeed, over the years he gave four different proofs of it. The theorem concerns a polynomial expression

> $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_{n-1} z + a_n$, where n is a positive integer, z a (typically complex) variable, and the other symbols denote constants.

The theorem states that the expression $p(z)$ possesses at least one zero, i.e. there is at least one[2] value z_1 for which $p(z_1) = 0$.

Gauss's four proofs were published in 1799, 1815, 1816 and 1848. The fourth made explicit use of the geometric interpretation, but otherwise closely resembled the first. This has led many commentators to speculate that he had this approach very much in mind as early as 1799. In that case therefore Hamilton would not have been the first to envisage the complex numbers as pairs of real ones.

In fact we now know on other grounds that he was not. That honor rightly belongs to one Caspar Wessel (1745-1818). Wessel was not in fact a professional mathematician, but rather a surveyor and cartographer (map-maker). His sole foray into mathematics proper was a paper published in 1797 by the Royal Danish Academy of Sciences, but probably written some ten years before this. This was explicitly concerned with the nature of complex numbers and also explicitly considered them as pairs of reals. As this publication clearly predates Hamilton's, the priority is clearly Wessel's.

As if this were not enough, there is yet another claimant with excellent credentials: Jean Argand (1768-1822). We have seen that Wessel was not a professional mathematician. Nor was Argand. He was an accountant. Nonetheless he published his geometrical interpretation of complex numbers in a privately published booklet printed in 1806. He also left his name off the work, which was thus completely anonymous. However a copy was sent to the eminent mathematician Adrien-Marie Legendre (1752-1833), who in turn passed it on to François Français (1768-1810). With Français' death, the booklet came into the hands of his brother Jacques (1775-1833). Jacques was so taken with the idea that he published an article based on it in one of the leading mathematical journals of the day. Jacques stated that his work was based on one by an unknown author. In consequence, Argand came out of hiding and revealed himself as the author in question. A paper based on his earlier work appeared in the same specialist journal, and so Argand was accorded due credit.[3]

Nonetheless, the work of Wessel and Argand went largely ignored, probably because of their amateur status. Wessel's work had appeared in the mainstream mathematical literature; nonetheless nobody had noticed. That Argand's initial anonymous

[2] A ready corollary tells us that there are in fact exactly n such zeroes (although in certain unusual cases some may be equal to others).

[3] Argand went on to other mathematical researches, most notably a proof of the Fundamental Theorem of Algebra.

booklet made no waves is understandable, but the work of Jacques Français should have rectified this. Argand's name now attaches to the plane on which the complex numbers are represented: the *complex plane* or *Argand Diagram*. Wessel is accorded no such honor.

There is a further property of the complex numbers that needs attention. It is one that we certainly would have taken for granted during our school years, but it is actually quite surprising, and it needs to be proved. It is absolutely vital if we are to use what in effect the algebra of the reals in in this extended context. That is to say, it is absolutely vital if the teacher's approach is to succeed.

It goes like this.

> If z_1 is one complex number, and z_2 is another, and if $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$.

We are so used to the truth of this proposition when asserted of the reals, that we are apt simply to assume without question that it applies to the complex numbers. Actually it is a quite surprising result. In technical language, it asserts that the complex numbers form a *division algebra*. Division algebras actually constitute a very rare species. The quaternions (based on sets of four real numbers)[4] give another case but only at the cost of abandoning the commutative law of multiplication (i.e. if Q_1 is one quaternion and Q_2 another then it will not in general be true that $Q_1Q_2 = Q_2Q_1$). Then there are various systems of *octonions* (based on sets of eight real numbers) which are also classed as division algebras, but here we have to abandon as well the associative law of multiplication (i.e. if O_1 is one octonion, O_2 another and O_3 a third, then it will not in general be true that $O_1(O_2O_3) = (O_1O_2)O_3$). These are all the division algebras there are. Thus the complex numbers constitute the *only* division algebra (other than the reals) for which all the usual laws of arithmetic apply.

The proof of the theorem just stated is not particularly difficult, but it isn't trivial either. Here is how it goes.

> Put $z_1 = \alpha + i\beta$ and $z_2 = \gamma + i\delta$. Then we need to show that $(\alpha + i\beta)(\gamma + i\delta) = 0$ implies either $\alpha = \beta = 0$ or else $\gamma = \delta = 0$. Expanding the product and equating both the real and the imaginary parts to zero yields two equations:
>
> $$\alpha\gamma - \beta\delta = 0$$
> $$\beta\gamma + \alpha\delta = 0.$$
>
> There are two possibilities: $\alpha = 0$ and $\alpha \neq 0$. First consider the case $\alpha = 0$. This implies that either $\beta = 0$ or $\delta = 0$ and also either $\beta = 0$ or $\gamma = 0$. If $\beta = 0$, we are done, because in that case $z_1 = 0$. Or else we could have $\gamma = \delta = 0$, in which case $z_2 = 0$. On the other hand if $\alpha \neq 0$, we may divide by it to find $\gamma = \beta\delta/\alpha$, and so deduce that $(\beta^2 + \alpha^2)\,\delta/\alpha = 0$. In order to satisfy this equation, we need either $\alpha = \beta = 0$, which cannot be since we assumed that $\alpha \neq 0$, or else $\delta = 0$, which implies that $\gamma = 0$, so that $z_2 = 0$.

[4] Quaternions were discussed at some length in my *Function* columns for June 1994 and October 1995.

This completes the proof, and as was mentioned above the result is somewhat surprising. Nonetheless, it is something we might well consider as obvious and not for a moment in doubt. Certainly our teacher's remark, "there's not really a problem; you just invent a square root, $\sqrt{-1} = i$, and proceed as if nothing untoward has happened", could be seen as predisposing one to accept the result without question.

However, in a sense, it is this property that justifies the teacher's approach. Without it, the properties of the complex numbers would be very different from those of the reals. The proof itself does not depend in any way on whether we use the teacher's approach or the textbook's; I have given it as the teacher would have, but I could quite easily have given it from the other perspective. In fact, the proof itself would hardly alter.

So we are brought back to the question my fellow students and I asked all those years ago: "What *is* the point of all this rigmarole?". The answer is subtle, and almost philosophical as much as strictly mathematical. What the "rigmarole" tells us is that the imaginary and complex numbers have exactly the same claim to existence as the reals. It is *exactly* the "rigmarole" that justifies the teacher's position. Without it, we may proceed by "just invent[ing] a square root, $\sqrt{-1} = i$, and proceed[ing] as if nothing untoward has happened", but we really have no notion of *why* this piece of legerdemain actually works.

However, I will close by quoting the final words of another textbook writer, H. A. Thurston in his book *The Number System*,

> "If we ignore the distinction between a real complex number [i.e. $[x, 0]$] and the real number corresponding to it [i.e. x] ... it will follow ... that each complex number is of the form $x + iy$, where x and y are real numbers and $i^2 = -1$. Moreover the definitions of multiplication and addition are what would be obtained by writing $[a, b]$ as $a + ib$ and so on, and multiplying out as though i were a real number, and then replacing i^2 by -1 wherever it occurs. It follows that every calculation with complex numbers can be carried out by this process, and that we can treat complex numbers from the usual elementary point of view."

In other words, the teacher's subterfuge works fine!