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Roots and all: An economical algorithm

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Every polynomial of degree n can be expressed as a product of factors,

$$\begin{aligned} p(x) &= a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \\ &= a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n). \end{aligned}$$

Note that the α_i, which may be real or complex, are the roots of the polynomial, i.e., $p(\alpha_i) = 0$ for all $i = 1, 2, \ldots, n$. It is natural to look for relationships between the roots of the polynomial α_i and the coefficients a_i. Two well-known relations are

$$\sum_{i=1}^{n} \alpha_i = -\frac{a_1}{a_0}$$

and

$$\prod_{i=1}^{n} \alpha_i = (-1)^n \frac{a_n}{a_0}.$$

A polynomial with $a_0 = 1$ is called a monic polynomial and in this case we can write

$$\begin{aligned} p(x) &= \prod_{i=1}^{n}(x-\alpha_i) \\ &= \sum_{i=0}^{n}(-1)^i \omega_i x^{n-i} \\ &= \omega_0 x^n - \omega_1 x^{n-1} + \omega_2 x^{n-2} - \cdots \end{aligned}$$

where

$$\begin{aligned} \omega_0 &= 1, \\ \omega_1 &= \sum_{j} \alpha_j, \\ \omega_2 &= \sum_{j<k} \alpha_j\alpha_k, \\ \omega_3 &= \sum_{j<k<l} \alpha_j\alpha_k\alpha_l, \\ &\vdots \end{aligned}$$

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It is possible to find identities relating the sums of the powers of the roots defined by

$$s_i = \sum_j (\alpha_j)^i, \quad i = 1, 2, \ldots,$$

to the coefficients, ω_j, of the monic polynomial, without actually finding the roots α_k explicitly. These identities are known as Newton identities, or Newton–Girard formulae.

It is straightforward to find recursion relations linking ω_i and s_i. Note that

$$\begin{aligned}\sum_{j=1}^{n} p(\alpha_j) &= \sum_{i=0}^{n} (-1)^i \omega_i \sum_{j=1}^{n} \alpha_j^{n-j} \\ &= \sum_{i=0}^{n} (-1)^i \omega_i s_{n-i} \\ &= 0,\end{aligned}$$

where $\omega_0 = 1$, $s_0 = n$. We now separate off the $i = 0$ contribution in the above sums and solve separately for s_n and ω_n to arrive at

$$s_n = \sum_{i=1}^{n} (-1)^{i-1} \omega_i s_{n-i} \tag{1}$$

and

$$\omega_n = \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1} s_i \omega_{n-i}. \tag{2}$$

We can now use the recursion relations to obtain s_i as functions of $\omega_1, \omega_2, \ldots, \omega_i$; and ω_i as functions of $s_1, s_2, \ldots, s_i$. The first few terms are as follows:

$$\begin{aligned} s_1 &= \omega_1 \\ s_2 &= \omega_1^2 - 2\omega_2 \\ s_3 &= \omega_1^3 - 3\omega_1\omega_2 + 3\omega_3 \\ s_4 &= \omega_1^4 - 4\omega_1^2\omega_2 + 4\omega_1\omega_3 + 2\omega_2^2 - 4\omega_4 \\ &\vdots \end{aligned}$$

and

$$
\begin{aligned}
\omega_1 &= s_1 \\
\omega_2 &= \frac{s_1^2}{2} - \frac{s_2}{2} \\
\omega_3 &= \frac{s_1^3}{6} - \frac{s_1 s_2}{2} + \frac{s_3}{3} \\
\omega_4 &= \frac{s_1^4}{24} - \frac{s_1^2 s_2}{4} + \frac{s_1 s_3}{3} + \frac{s_2^2}{8} - \frac{s_4}{4} \\
&\vdots
\end{aligned}
$$

By proceeding recursively in this fashion we can obtain ω_i as functions of the s_1, s_2, $\ldots, s_i$ for any finite i, but the process is laborious. We now describe a more economical way to obtain these expressions for ω_i.

In general we can write

$$
\begin{aligned}
s_n &= s_n(\omega_1, \omega_2, \ldots, \omega_n) \\
&= \frac{1}{k^n} s_n(k\omega_1, k^2\omega_2, \ldots, k^n\omega_n),
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_n &= \omega_n(s_1, s_2, \ldots, s_n) \\
&= \frac{1}{k^n} \omega_n(ks_1, k^2 s_2, \ldots, k^2 s_n),
\end{aligned}
$$

where $k \neq 0$. We show that the coefficients in $s_n(\omega_1, \omega_2, \ldots, \omega_n)$ and $\omega_n(s_1, s_2, \ldots, s_n)$ can be determined by solving a system of linear equations as follows:

The selection $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ is such that $s_i = n$, $\omega_i = \binom{n}{i}$ $i = 1, 2, \ldots, n$, hence

$$
\begin{aligned}
s_n\left(\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\right) &= n, \\
\omega_n(n, n, \ldots, n) &= 1.
\end{aligned}
$$

Since the number of roots assigned have no bearing on the outcome (they are merely passengers), the equations above can be extended to the identities

$$
s_n\left(\binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{n}\right) \equiv j, \quad \forall j \tag{3}
$$

$$
\omega_n[j] := \omega_n(j, j, \ldots, j) \equiv \binom{j}{n} \tag{4}
$$

where $\binom{j}{n} = 0$ if $j < n$.

For example,

$$\omega_3(s_1, s_2, s_3) = as_1^3 + bs_1s_2 + cs_3,$$

where a, b, c satisfy the equations

$$\begin{aligned}
\omega_3[1] &= a + b + c = \binom{1}{3} = 0, \\
\frac{1}{2}\omega_3[2] &= 4a + 2b + c = \frac{1}{2}\binom{2}{3} = 0, \\
\frac{1}{3}\omega_3[3] &= 9a + 3b + c = \frac{1}{3}\binom{3}{3} = \frac{1}{3}.
\end{aligned}$$

The number of terms in $s_n(\omega_1, \omega_2, \ldots, \omega_n)$ and $\omega_n(s_1, s_2, \ldots, s_n)$ is the partition function $\Pi(n)$ of n, namely the number of integer solutions $(x_1, x_2, \ldots, x_n)$, $x_k \geq 0$ of the equation

$$\sum_{k=1}^{n} kx_k = n.$$

An alternate algorithm to (2) for ω_n follows:
Define $\omega_n^{(i)}$, $i = 1, 2, \ldots, n-1$ by

$$\begin{aligned}
\omega_n^{(1)} &= \omega_n(0, s_2, s_3, \ldots, s_n) \\
\omega_n^{(2)} &= \omega_n(0, 0, s_3, \ldots, s_n) \\
\omega_n^{(3)} &= \omega_n(0, 0, 0, s_4, \ldots, s_n) \\
&\vdots
\end{aligned}$$

so that the number of terms in $\omega_n^{(i)}$ is the i-th partition function $\Pi^{(i)}(n)$ of n, namely the number of integer solutions $(x_{i+1}, x_{i+2}, \ldots, x_n)$, $x_k \geq 0$ of

$$\sum_{k=i+1}^{n} kx_k = n$$

and it satisfies the difference equation

$$\Pi^{(i)}(n) - \Pi^{(i-1)}(n) = -\Pi^{(i-1)}(n-i), \quad \Pi^{(0)}(n) = \Pi(n) \tag{5}$$

leading to

$$\Pi^{(i)}(n) = \begin{cases} \sum_{k=i+1}^{\lfloor \frac{n}{2} \rfloor} \Pi^{(k-1)}(n-k) + 1 & \text{for} \quad 0 \leq i \leq \lfloor \frac{n}{2} - 1 \rfloor \\ 1 & \text{for} \quad \lfloor \frac{n}{2} \rfloor \leq i \leq n-1, \end{cases} \tag{6}$$

where $\lfloor \frac{n}{2} \rfloor$ is the largest integer smaller or equal to $\frac{n}{2}$, since $\Pi^{(k-1)}(n-k) = 0$ for $k > \lfloor \frac{n}{2} \rfloor$.

Moreover, the partial derivatives of $\omega_n^{(i)}$ with respect to $s_1, s_2, \ldots, s_n$ satisfy the canonical relations

$$\frac{\partial \omega_n^{(i)}}{\partial s_k} = \begin{cases} \frac{(-1)^{k-1}}{k}\omega_{n-k}^{(i)} & \text{for } 0 \le i \le \lfloor \frac{n}{2} - 1 \rfloor,\ i+1 \le k \le n-i-1 \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $\omega_n^{(0)} = \omega_n$. Lastly, integrating (7) in accordance with (6) yields

$$\omega_n^{(i)} = \sum_{k=i+1}^{\lfloor \frac{n}{2} \rfloor} \int \frac{(-1)^{k-1}}{k} \omega_{n-k}^{(k-1)} \partial s_k + \frac{(-1)^{n-1}}{n} s_n, \quad i = 0, 1, \ldots, n-1 \tag{8}$$

where the sought after algorithm for ω_n corresponds to $i = 0$.

Examples:

(i)

$$\omega_4^{(1)} = \int -\frac{1}{2}\omega_2^{(1)} \partial s_2 - \frac{s_4}{4}$$
$$\omega_2^{(1)} = -\frac{s_2}{2} \therefore \int -\frac{1}{2}\omega_2^{(1)} \partial s_2 = \frac{s_2^2}{8}.$$

(ii)

$$\omega_5 = \int \omega_4 \partial s_1 + \int -\frac{1}{2}\omega_3^{(1)} \partial s_2 + \frac{s_5}{5}$$
$$\int \omega_4 \partial s_1 = \frac{s_1^5}{120} - \frac{s_1^3 s_2}{12} + \frac{s_1^2 s_3}{6} + \frac{s_1 s_2^2}{8} - \frac{s_1 s_4}{4}$$
$$\omega_3^{(1)} = \frac{s_3}{3} \therefore \int -\frac{1}{2}\omega_3^{(1)} \partial s_2 = -\frac{s_2 s_3}{6}.$$

(iii)

$$\omega_6 = \int \omega_5 \partial s_1 + \int -\frac{1}{2}\omega_4^{(1)} \partial s_2 + \int \frac{1}{3}\omega_3^{(2)} \partial s_3 - \frac{s_6}{6}$$
$$\int \omega_5 \partial s_1 = \frac{s_1^6}{720} - \frac{s_1^4 s_2}{48} + \frac{s_1^3 s_3}{18} + \frac{s_1^2 s_2^2}{16} - \frac{s_1^2 s_4}{8} - \frac{s_1 s_2 s_3}{6} + \frac{s_1 s_5}{5}$$
$$\omega_4^{(1)} = \frac{s_2^2}{8} - \frac{s_4}{4} \therefore \int -\frac{1}{2}\omega_4^{(1)} \partial s_2 = -\frac{s_2^3}{48} + \frac{s_2 s_4}{8}$$
$$\omega_3^{(2)} = \frac{s_3}{3} \therefore \int \frac{1}{3}\omega_3^{(2)} \partial s_3 = \frac{s_3^2}{18}.$$

The latter is less exhausting than (2) with $n = 6$, which carries a surplus of $\sum_{i=0}^{5} \Pi(i) - \Pi(6) = 1+1+2+3+5+7-11 = 8$ like terms to the ones allocated. This surplus grows exponentially with n: Possibly an incentive to go economically?