Parabola Volume 50, Issue 2 (2014)

Solutions 1440–1450

We begin with the solution to problem 1440 from volume 49, issue 3, which was inadvertently omitted last issue.

Q1440 Let $f(x)$ be a polynomial with degree 2012, such that

$$f(1) = 1\ , \quad f(2) = \frac{1}{2}\ , \quad f(3) = \frac{1}{3}\ , \ldots , \quad f(2013) = \frac{1}{2013}\ .$$

Find the value of $f(2014)$.

SOLUTION We can write the given equations as

$$xf(x) - 1 = 0$$

for $x = 1, 2, 3, \ldots, 2013$. Since $xf(x) - 1$ is a polynomial with degree 2013 which has 2013 roots, we can give its factorisation, involving one unknown constant:

$$xf(x) - 1 = c(x-1)(x-2)\cdots(x-2012)(x-2013)\ . \qquad (*)$$

By substituting $x = 0$ we obtain

$$-1 = c(-1)(-2)\cdots(-2012)(-2013)$$

and so $c = 1/2013!$. By substituting this value back into equation $(*)$ and taking $x = 2014$ we find

$$2014f(2014) - 1 = \frac{1}{2013!}(2013)(2012)\cdots(2)(1) = 1$$

and so

$$f(2014) = \frac{2}{2014} = \frac{1}{1007}\ .$$

Q1441 Hundreds of millions of pebbles were deposited on the foreshore of a beach. Every high tide, 20% of the pebbles are transported from the deep end to shallow end. Every low tide, 14% of the pebbles are transported from the shallow to the deep end. Over a long period of time, what is the exact fraction of the pebbles on the deep end after high tide?

SOLUTION We assume that the pebbles are infinitely divisible, so that regardless of their number it is possible to move *exactly* 20% or 14% of them. We shall also assume for the time being that the fraction of the pebbles on the deep end after high tide does settle down to some fixed number – this is perhaps not obvious, but we shall justify the assumption later.

Let the proportion of pebbles at the deep end after a certain high tide be x. The first thing that happens now is that there is a low tide, and 14% of the remaining fraction

$1-x$ of the pebbles are transferred to the deep end, which now accounts for $x+0.14(1-x)$ of the pebbles. At the next high tide, 20% of these are lost and the fraction remaining at the deep end after the high tide is $0.8(x+0.14(1-x))$. If this is the same as it was after the previous high tide we have

$$x = 0.8(x + 0.14(1 - x)) \ ,$$

an equation which simplifies to

$$x = 0.112 + 0.688x$$

and is easily solved to give $x = \frac{14}{39}$.

We have proved that **if** the fraction of pebbles at the deep end after high tide settles down to a fixed number, then the number must be $\frac{14}{39}$. To show that this does actually happen, let x be the proportion after some high tide and x' the proportion after the next high tide. Exactly the same reasoning as in the previous paragraph shows that

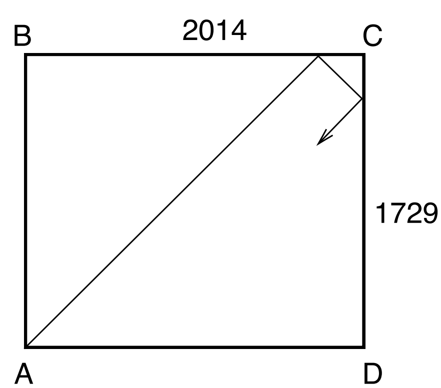
$$x' = 0.112 + 0.688x \ .$$

Some straightforward algebra enables us to rewrite this as

$$x' - \tfrac{14}{39} = 0.688(x - \tfrac{14}{39}) \ .$$

Therefore the difference between the fraction of interest and $\frac{14}{39}$ decreases after every successive pair of tides, and becomes closer and closer to zero as time goes on. This justifies our assumption that the fraction of pebbles at the deep end settles down to some fixed proportion, and shows that this proportion is $\frac{14}{39}$.

Q1442 $ABCD$ is a rectangle consisting of 2014×1729 squares. A particle is dispatched from A at $45°$ and bounces off BC then CD, and so on. If it reaches one of the corners A, B, C, D it falls through a tiny hole and leaves the rectangle. Will this ever happen? If so, which corner will the particle fall through, and how many walls will it hit during its journey?



SOLUTION The total distance travelled *horizontally* by the particle when it hits the left or right wall for the mth time is $2014m$. The total distance travelled *vertically* when it hits the top or bottom wall for the nth time is $1729n$. The particle reaches a corner when it hits a left or right wall simultaneously with a top or bottom wall. But since the particle is moving at a $45°$ angle, the horizontal and vertical distances must always be the same and we therefore have

$$2014m = 1729n\ .$$

We can cancel 19 to give

$$106m = 91n\ ;$$

since 106 and 91 have no common factor, the smallest possible solution is $m = 91$, $n = 106$. That is to say, the particle is hitting a left or right wall for the 91st time, and the top or bottom wall for the 106th. Since m is odd the particle is in fact hitting the right wall, and since n is even it is hitting the bottom wall: that is, the first (and therefore only) corner reached by the particle is D. The number of walls it hits before this happens is $(m-1)+(n-1) = 195$.

NOW TRY problem 1452.

Q1443

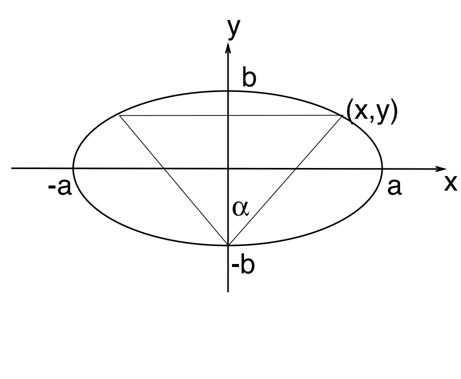
(a) Factorise the quartic $x^4 + 6x^2 - 16x + 9$.

(b) Use the previous result to find the maximum value of $\dfrac{x}{(x^2+3)^2}$.

(c) Find the maximum area of a triangle inscribed within the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as shown: one vertex of the triangle is to be at the point $(0, -b)$ and the opposite side is to be horizontal.



SOLUTION

(a) It is not hard to see that $x = 1$ is a root of the quartic, and by long (or short) division we have

$$x^4 + 6x^2 - 16x + 9 = (x-1)(x^3 + x^2 + 7x - 9)\ .$$

And now $x = 1$ is also a root of the cubic, so

$$x^4 + 6x^2 - 16x + 9 = (x-1)^2(x^2 + 2x + 9) .$$

This is the complete factorisation because the quadratic has no roots: to see this, either calculate the discriminant and observe that it is negative, or complete the square to see that

$$x^2 + 2x + 9 = (x+1)^2 + 8 > 0 .$$

Comment. To state this more precisely, the quadratic has no *real* roots. If you have learned about complex numbers you could use them to take the factorisation one step further.

(b) From (a), we have

$$x^4 + 6x^2 - 16x + 9 \geq 0$$

for all real values of x. This can be written as

$$(x^2 + 3)^2 \geq 16x$$

and so

$$\frac{x}{(x^2+3)^2} \leq \frac{1}{16} .$$

Thus the expression on the left hand side can never be greater than $\frac{1}{16}$; and it is equal to $\frac{1}{16}$ when $x = 1$; so this is the required maximum value.

Comment. If you have learned calculus, you could use differentiation to confirm this result.

(c) Let α be the angle shown, let (x, y) be the "top right" vertex of the triangle, and let A be the area of the triangle. Since the triangle has base length $2x$ and altitude $y + b$ we have

$$A = x(y + b) .$$

Now we need to find (x, y), the intersection point of the line and ellipse, in terms of α. The line through $(0, -b)$ and (x, y) has gradient $\tan(\frac{\pi}{2} - \alpha) = \cot\alpha$ and so its equation is

$$y = x\cot\alpha - b .$$

Substituting into the equation of the ellipse and simplifying,

$$\frac{x^2}{a^2} + \frac{(x\cot\alpha - b)^2}{b^2} = 1$$

and so

$$b^2x^2 + a^2(x^2\cot^2\alpha - 2bx\cot\alpha + b^2) = a^2b^2$$

and so

$$(b^2 + a^2\cot^2\alpha)x^2 - 2a^2bx\cot\alpha = 0 .$$

Now this equation has a solution $x = 0$ which gives the intersection point $(0, -b)$, which is not the one we want; we need the other solution

$$x = \frac{2a^2 b \cot\alpha}{b^2 + a^2 \cot^2\alpha} .$$

Therefore

$$A = x(y+b) = x^2 \cot\alpha = \frac{4a^4 b^2 \cot^3\alpha}{(b^2 + a^2\cot^2\alpha)^2} .$$

Multiplying top and bottom by $\tan^4\alpha$ gives a slightly simpler expression

$$A = \frac{4a^4 b^2 \tan\alpha}{(b^2\tan^2\alpha + a^2)^2} ,$$

and if we write

$$z = \frac{b}{a}\sqrt{3}\,\tan\alpha$$

this becomes

$$A = 12\sqrt{3}\,ab\,\frac{z}{(z^2+3)^2}$$

But now recall the result of (b). The quotient $z/(z^2+3)^2$ has a maximum value when $z = 1$, that is, $\tan\alpha = a/b\sqrt{3}$, and this maximum value is $\frac{1}{16}$. So the maximum possible area for the triangle is

$$A = \frac{12\sqrt{3}\,ab}{16} = \frac{3\sqrt{3}}{4}\,ab .$$

Alternative solution, for those who know about parametrically defined curves and calculus. Since (x, y) is on the ellipse we have $x = a\cos\theta$ and $y = b\sin\theta$ for some θ, and then the area of the triangle is

$$A = x(y+b) = ab\cos\theta(1+\sin\theta) .$$

Using standard techniques – readers are invited to fill in the details – this has a maximum when $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$, and so the greatest possible area is

$$A = ab\,\frac{\sqrt{3}}{2}\,\frac{3}{2} = \frac{3\sqrt{3}}{4}\,ab$$

as above.

NOW TRY problem 1451.

Q1444 In a certain town each of the inhabitants is either a truth–teller or a liar; however this does not mean that everyone is actually able to answer every question they are asked. If a truth–teller is absolutely certain of the answer to a question, he will give that answer; if not, he will say, "I don't know." On the other hand, a liar will never truthfully admit to not knowing something: he will give an answer that he knows is false, if any, but if there is nothing that he is *certain* is false then he will give a randomly chosen answer (possibly even, by accident, the true answer). Moreover, everyone in this town can instantly deduce the logical consequences of any facts they know.

I meet four inhabitants of this town and ask them, "How many of you four are truth–tellers?"

Kevin says, "I don't know"; then Laura says, "One"; then Mike says, "None." Noela, however, is asleep. Fortunately I don't need to wake her up, since I can already tell whether she is a truth–teller or a liar. Which?

SOLUTION Kevin is certainly a truth–teller, as a liar will never admit, "I don't know." If Laura were a truth–teller then, having heard Kevin's answer, she would know that he was a truth–teller too, and would not give the false answer "One"; so she must be a liar. Mike must be a liar for the same reason. Now suppose that Noela is a liar. Then Laura would have given the right answer: as she is a liar, the only way this could happen is if she didn't know any *definitely* false answer. But this is not the case: she knew that Kevin was a truth–teller and therefore that "None" would have been a false answer. It must therefore be that Noela is a truth–teller (and that Laura knew this, and therefore knew that "One" was a false answer).

NOW TRY problem 1453.

Q1445 The *Lucas sequence*, a famous sequence of numbers closely related to the even more famous Fibonacci sequence, is

$$1,\ 3,\ 4,\ 7,\ 11,\ 18,\ldots\ :$$

every number in the sequence, apart from the first two, is the sum of the previous two numbers. Prove that if we take every second number in the Lucas sequence, that is, $3, 7, 18, \ldots$ and so on for ever, we will never find a number which is a perfect square.

SOLUTION Let's continue the Lucas sequence a bit further,

$$1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322,\ 521,\ 843,\ldots\ ,$$

and write down the *last digit* of each of these numbers:

$$1,\ 3,\ 4,\ 7,\ 1,\ 8,\ 9,\ 7,\ 6,\ 3,\ 9,\ 2,\ 1,\ 3,\ldots\ .$$

Notice that the last two numbers we have written down are the same as the first two: and since each subsequent term depends only on the previous two, the sequence of last digits will now repeat. That is, the *complete* sequence of last digits will be

$$1,\ 3,\ 4,\ 7,\ 1,\ 8,\ 9,\ 7,\ 6,\ 3,\ 9,\ 2,$$

repeated over and over. If we now extract every second digit we get

$$3,\ 7,\ 8,\ 7,\ 3,\ 2,$$

once again repeated indefinitely. However, if we take a number which has last digit $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, then its square will have last digit $0, 1, 4, 9, 6, 5, 6, 9, 4, 1$ respectively. Therefore the numbers we are looking at, which have last digit $2, 3, 7$ or 8, can never be squares.

Q1446 Use the Peano axioms for the natural numbers (see Michael Deakin's article in this issue) to prove the cancellation law for addition: for all numbers a, b, c, if $a + c = b + c$ then $a = b$. The basic facts you may use are the five axioms, Michael's equations (1) and (2) which define addition, and *nothing else*. In particular, you cannot just say "subtract c from both sides", because subtraction has not yet been defined!

SOLUTION We prove the result by induction. Firstly take $c = 1$. Let a and b be any numbers, and suppose that $a + c = b + c$. That is, $a + 1 = b + 1$; from equation (1) we have $a^+ = b^+$. But now axiom 4 tells us that $a = b$. Therefore the result is true in the case $c = 1$.

Now suppose the result true for some specific c: that is, for this particular c and for any a and b, if $a + c = b + c$ then $a = b$. We have to prove that for any a and b, if $a + c^+ = b + c^+$ then $a = b$. So, suppose that $a + c^+ = b + c^+$; using equation (2), we have $(a + c)^+ = (b + c)^+$. Once again axiom 4 tells us that $a + c = b + c$; and by the inductive assumption, $a = b$. This proves the inductive step, and shows that the result is true for all values of c.

NOW TRY problem 1459.

Q1447 If the roots of the equation $x^3 - x + 2014 = 0$ are $\tan\alpha,\ \tan\beta$ and $\tan\gamma$, evaluate $\tan(\alpha + \beta + \gamma)$.

SOLUTION Using the formula for the tangent of a sum, we have

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

and so

$$\begin{aligned}
\tan(\alpha + \beta + \gamma) &= \tan((\alpha + \beta) + \gamma) \\
&= \frac{\tan(\alpha + \beta) + \tan\gamma}{1 - \tan(\alpha + \beta)\tan\gamma} \\
&= \frac{\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} + \tan\gamma}{1 - \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\ \tan\gamma} \\
&= \frac{(\tan\alpha + \tan\beta + \tan\gamma) - (\tan\alpha\tan\beta\tan\gamma)}{1 - (\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha)}\ .
\end{aligned}$$

However, from the relations between the coefficients and roots of a polynomial, we have

$$\tan\alpha + \tan\beta + \tan\gamma = 0$$
$$\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = -1$$
$$\tan\alpha\tan\beta\tan\gamma = -2014$$

and hence

$$\tan(\alpha+\beta+\gamma) = \frac{0-(-2014)}{1-(-1)} = 1007\ .$$

NOW TRY problem 1454.

Q1448 Let $f(x)$ be a polynomial with degree 2012, such that

$$f(1) = 1\ , \quad f(2) = \frac{1}{2}\ , \quad f(3) = \frac{1}{3}\ , \ldots , \quad f(2013) = \frac{1}{2013}\ .$$

Find the value of $f(0)$.

SOLUTION As in the solution of problem 1440 (see the previous issue) we have

$$xf(x) - 1 = c(x-1)(x-2)\cdots(x-2012)(x-2013) \qquad (*)$$

with $c = 1/2013!$. Now $f(0)$ is the constant term in $f(x)$; it is therefore the coefficient of x in $xf(x)$, which is also the coefficient of x in the right hand side of $(*)$. If we expand the product we will get terms

$$c(x)(-2)(-3)(-4)\cdots(-2013) = \frac{2\times3\times4\times\cdots\times2013}{1\times2\times3\times4\times\cdots\times2013} = 1$$

and

$$c(-1)(x)(-3)(-4)\cdots(-2013) = \frac{1\times3\times4\times\cdots\times2013}{1\times2\times3\times4\times\cdots\times2013} = \frac{1}{2}$$

and

$$c(-1)(-2)(x)(-4)\cdots(-2013) = \frac{1\times2\times4\times\cdots\times2013}{1\times2\times3\times4\times\cdots\times2013} = \frac{1}{3}$$

and so on; the total coefficient of x will be

$$f(0) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2013}\ .$$

Q1449

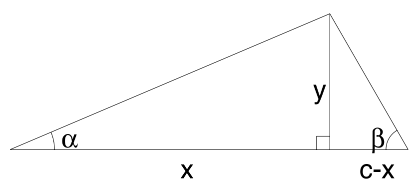
(a) A triangle has angles α and β, the side between these two angles being of length c. Show that the area of the triangle is given by

$$A = \frac{1}{2}c^2\,\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\ .$$

(b) The square $ABCD$ has side length 1. A line is drawn through A intersecting BC; it makes an angle α with the diagonal through A. A line is drawn through B intersecting CD; it makes an angle β with the diagonal through B. Lines are drawn through C and D parallel to those through A and B respectively. If the parallelogram determined by these four lines is removed from the square, find the remaining area.

SOLUTION

(a) Label lengths and angles in the triangle as shown.



Writing down two expressions for the altitude, we have

$$y = x \tan\alpha \quad \text{and} \quad y = (c - x)\tan\beta\ .$$

Equating these and solving for x yields

$$x = \frac{c\tan\beta}{\tan\alpha + \tan\beta}$$

and hence

$$y = \frac{c\tan\alpha\tan\beta}{\tan\alpha + \tan\beta}\ .$$

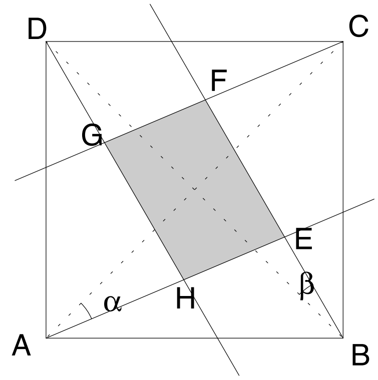
We can simplify this by multiplying numerator and denominator by $\cos\alpha\cos\beta$, giving

$$y = \frac{c\sin\alpha\sin\beta}{\sin\alpha\cos\beta + \sin\beta\cos\alpha} = \frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\ .$$

The area of the triangle is therefore

$$A = \frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}cy = \frac{1}{2}\,c^2\,\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\ .$$

(b) The area we wish to calculate consists of two triangles like $\triangle AEB$ and two like $\triangle BFC$ as shown in the figure.



Using the above formula with side length $c = 1$, and carefully noting the angles involved, we have

$$\begin{aligned}
\text{area} &= \frac{\sin(45^\circ - \alpha)\sin(45^\circ + \beta)}{\sin(90^\circ - \alpha + \beta)} + \frac{\sin(45^\circ - \beta)\sin(45^\circ + \alpha)}{\sin(90^\circ + \alpha - \beta)} \\
&= \frac{\cos(45^\circ + \alpha)\cos(45^\circ - \beta)}{\cos(\alpha - \beta)} + \frac{\sin(45^\circ + \alpha)\sin(45^\circ - \beta)}{\cos(\alpha - \beta)} \\
&= \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} \, .
\end{aligned}$$

Q1450 Adam, Betty, Cathy, Doug, Ellen, Fiona, George and Harry have 10 coins, all of equal value, and are going to share them by the following procedure.

- Harry will propose a distribution of the coins (so many to himself, so many to George and so on). The other seven will vote on it. If 50% or more votes are in favour of Harry's proposal it is accepted and the coins are distributed accordingly.
- If Harry's proposal is rejected then he receives no coins and goes home. In this case George makes a proposal which is voted upon by the other six. Once again, if 50% or more are in favour then George's proposal is accepted; if not, George goes home with nothing.
- As long as proposals are rejected, the procedure continues, with one person fewer each time. If, eventually, Betty has a proposal rejected, then Adam gets all the coins.

How many coins will each person receive? We assume that all the participants are rational, and that they know the others are rational too. "Rational" means that they are able to work out the precise consequences of their actions; and that a person proposing a distribution of the coins will make the proposal which will bring him the maximum possible number of coins; and that each person will vote against a proposal if doing so *cannot* give him a worse outcome than voting for it, but will vote for it otherwise. If a person making a proposal for the distribution of the coins has two or more options which are equally good for him, then he will choose between them at random, for example by writing them down on slips of paper and pulling one out of a hat.

SOLUTION As in the previous solution, if it falls to Ellen to make a proposal for distribution of the coins she will suggest $7, 0, 1, 0, 2$ respectively to herself, Doug, Cathy, Betty and Adam; and this will be accepted because a majority will realise that they will do worse if it is rejected. (Please re–read the previous issue if you don't remeber why!)

Now Fiona thinks: I know what Ellen is thinking. To get my proposal accepted I have to get three voters onside; to do this with the smallest possible number of coins I will offer 1 to Doug, 2 to Cathy and 1 to Betty, leaving 6 for myself. So if it comes to me, I shall propose $6, 0, 1, 2, 1, 0$ to myself and the next five people.

George thinks: I know what Fiona is thinking. To avoid missing out I need three votes. So I will offer 1 coin each to Ellen and Adam; to get a third vote I will offer 2 coins to either Doug or Betty – I don't care which. So I can propose either $6, 0, 1, 2, 0, 0, 1$ or $6, 0, 1, 0, 0, 2, 1$; I will choose one of them by tossing a coin. Whichever proposal I make, three people will vote for it and it will be accepted.

Harry thinks: I know what George is thinking. He is going to toss a coin between two possibilities: either Doug or Betty will receive 2 coins, the other will get none, but they cannot possibly know who gets the money as it depends on chance. So if my proposal is defeated, the amounts that people can count on receiving are $6, 0, 1, 0, 0, 0, 1$. In fact, either Doug or Betty will receive 2 coins, but neither of them individually can be *certain* of this outcome. I will therefore offer 1 coin each to Fiona, Doug, Cathy and Betty, leaving 6 for myself. Fiona and Cathy will vote for this because if it is defeated they will do worse; Doug and Betty will vote for it because if it is defeated they *might* do worse.

Harry will make the proposal he has just decided upon, and Fiona, Doug, Cathy and Betty will vote for it for the reasons just given. Therefore the outcome is: 6 coins to Harry; 1 each to Fiona, Doug, Cathy and Betty; none to George, Ellen and Adam.

NOW TRY problem 1460.