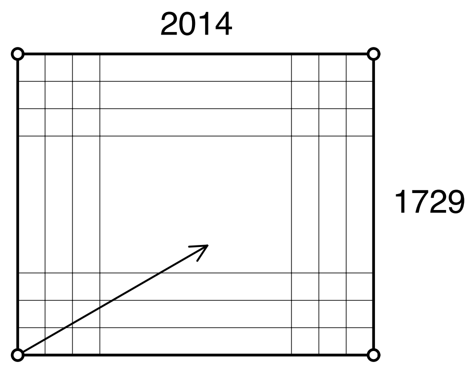
Parabola Volume 50, Issue 3 (2014)

Problems 1461–1470

Parabola incorporating Function would like to thank Sin Keong Tong for contributing problem 1464.

Q1461 As in problems 1442 and 1452, a particle is projected from one corner of a 2014×1729 rectangle. This time, however, the particle is projected at an angle of $30°$ above the horizontal. Find which of the four corners, if any, the particle eventually reaches.



Q1462 As expected, Mitchell won his dice game with Dale (see problem 1457, solution later this issue). Dale says, "Well of course you had the advantage: you had to get a 1 from a five–sided die and I had to do it from a six–sided die. Not only that, but you got to throw first. Let's play again, with me going first. Then we'll have equal chances." Mitchell replies, "OK, fine, but to make it more interesting we'll both double the number of sides on our dice: I'll use a 10–sided die and you'll use one with 12 sides. You can start, and the first to throw a 1 will win. Obviously, you still have an even chance of winning."

(a) Show that Dale was right in saying that the two had even chances in the game he proposed.
(b) Should he accept Mitchell's alternative offer?

Q1463 (Compare problem 1456, solution later this issue.) The positive integers a and b have no common factor. The positive integer n is a multiple of both a and b; exactly half the numbers from 1 to n are multiples of a or of b **or both**. Find a and b.

Q1464 Suppose that the numbers $x_1, x_2, \ldots, x_{2014}$ form a geometric sequence; and that $y_1, y_2, \ldots, y_{2014}$ is a sequence in which $y_1 = 1$ and $y_{2014} = \frac{1}{2014}$. If

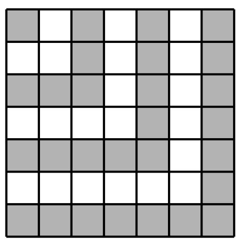
$$x_1^{y_1} = x_2^{y_2} = \cdots = x_{2014}^{y_{2014}},$$

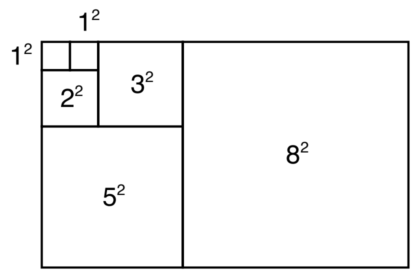
evaluate the sum

$$\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_{2014}} .$$

Q1465 Show that among any nine people as described in Theorem 3 of the article on "picture proofs" earlier this issue, there must always be four who pairwise like each other or three who pairwise dislike each other.

Q1466 (See the article on "picture proofs" earlier this issue.) What do the following diagrams prove?





Q1467 Find two different positive integers m and n such that

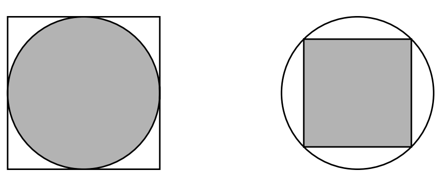
$$\lfloor 1000000\sqrt{m} \rfloor = \lfloor 1000000\sqrt{n} \rfloor .$$

The notation $\lfloor x \rfloor$ denotes the result of rounding x to the nearest integer downwards, for example, $\lfloor \pi \rfloor = 3$ and $\lfloor 4 \rfloor = 4$.

Q1468 Find all solutions of the simultaneous equations

$$x^2 - yz = 1 , \quad y^2 - zx = 2 , \quad z^2 - xy = -1 .$$

Q1469 The diagrams show a circle of radius a with a circumscribed square, and a circle of radius b with an inscribed square.



(a) Show that the ratio of the shaded area to the whole area for the first diagram is greater than $\frac{2}{3}$, and for the second diagram less than $\frac{2}{3}$.

(b) If the shaded area for both parts of the diagram together is exactly $\frac{2}{3}$ of the total area, find the ratio a/b.

(c) If the circles are the same size (that is, $a = b$) and we take m copies of the first diagram and n copies of the second diagram, is it possible to get the shaded area being $\frac{2}{3}$ of the total area?

Q1470 A triangle has sides a, b, c; the angles opposite these sides are A, B, C respectively. Prove that

$$a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0 \,.$$