Parabola Volume 60, Issue 1 (2024)

Numerically computed double, triple and quadruple planar bubbles for density r^p

Marcus Collins[1]

1 Introduction

The isoperimetric problem is one of the oldest in mathematics. It asks for the least-perimeter way to enclose given volume. For a single volume in Euclidean space (with uniform density) of any dimension, the well-known solution is any sphere. With density r^p, Boyer et al. [1] found that the solution for a single volume is a sphere *through* the origin. For *two* volumes in Euclidean space, Reichardt [11] showed that the standard double bubble, consisting of three spherical caps meeting along a sphere in threes at $120°$ angles as in Figure 1, provides an isoperimetric cluster. Hirsch et al. [7] conjectured that the isoperimetric cluster for two volumes in $\mathbb{R}^n$ with density r^p for $p > 0$ is the same Euclidean standard double bubble with a vertex at the origin, as in Figure 2, and showed that it is better for example than putting the centre at the origin. But it is not even known whether each region and the whole cluster are connected. As for the triple bubble, the minimizer in the plane with density r^p cannot just be the Euclidean minimizer [14] with central vertex at the origin, because the outer arcs do not have constant generalized curvature.

Hirsch et al. [7] proved existence, boundedness and regularity: a planar isoperimetric cluster consists of constant generalized-curvature curves meeting in threes at $120°$ (see our Proposition 4).

In this paper, we numerically compute double, triple and quadruple bubbles in the plane with density r^p for various $p > 0$, using Brakke's Evolver [2]. Some videos are available on Google Drive. Proposition 7 supports the conjecture of Hirsch et al. [7] that the optimal double bubble is the Euclidean one with one vertex at the origin (Figure 2). Proposition 8 indicates that the optimal triple bubble resembles the Euclidean one (Figure 1) with one vertex at the origin, but as p increases, one circular arc from the origin shrinks so that all arcs pass near the origin and remain approximately circular, as in Figure 3. (A constant-generalized-curvature curve is a circle if and only if it passes through the origin (Remark 6)). Proposition 10 indicates that the optimal quadruple bubble resembles the Euclidean one (Figure 1) with a vertex at the origin for small p (Figure 7), but as p increases to 1, one circular arc from the origin shrinks to a point and thereafter four arcs meet at the origin (Figure 8). This not does violate regularity because the density vanishes at the origin.

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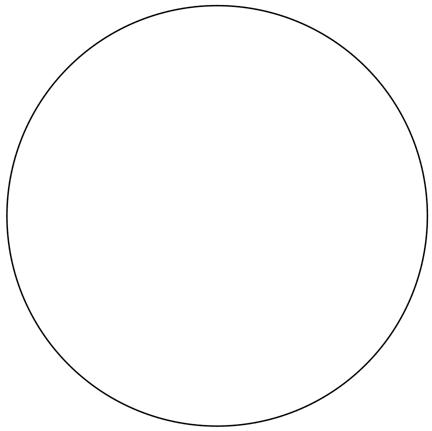
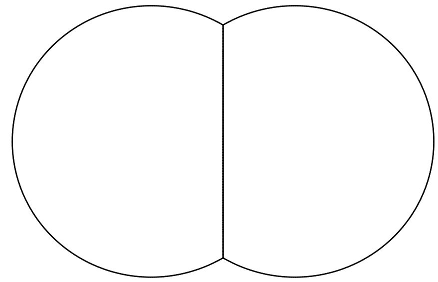
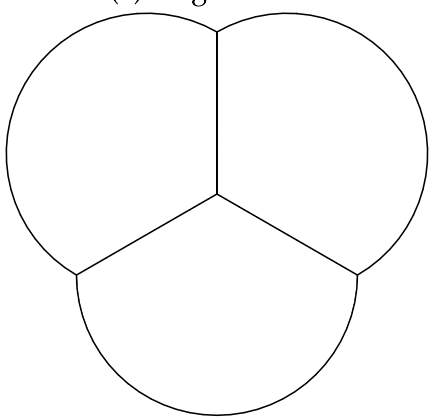


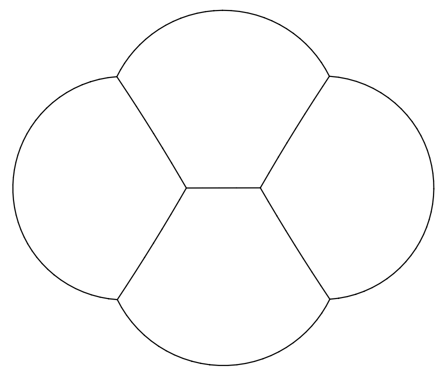
(a) Single bubble



(b) Double bubble



(c) Triple bubble



(d) Quadruple bubble

Figure 1: The optimal Euclidean single, double, triple and quadruple bubbles with equal areas.

History

Examination of isoperimetric regions in the plane with density r^p began in 2008 when Carroll et al. [3] showed that the isoperimetric solution for a single area in the plane with density r^p is a convex set containing the origin. It was something of a surprise when Dahlberg et al. [4] proved that the solution is a circle through the origin. In 2016, Boyer et al. [1] extended this result to higher dimensions. In 2019, Huang et al. [8] studied the 1-dimensional case, showing that the best single bubble is an interval with one endpoint at the origin and that the best double bubble is a pair of adjacent intervals which meet at the origin. Ross [13] showed that, in $\mathbb{R}^1$, multiple bubbles start with the two smallest meeting at the origin and the rest in increasing order alternating side to side.

2 Definitions

Definition 1 (Density Function)**.** Given a smooth Riemannian manifold M, a *density* on M is a positive continuous function (perhaps vanishing at isolated points) that weights each point p in M with a certain mass $f(p)$. Given a region $\Omega \subset M$ with piecewise smooth boundary, the weighted volume (or area) and boundary measure or perimeter of Ω are given by

$$V(\Omega) = \int_{\Omega} f \, \mathrm{d}V_0 \quad \text{and} \quad P(\Omega) = \int_{\partial\Omega} f \, \mathrm{d}P_0 \ ,$$

where $\mathrm{d}V_0$ and $\mathrm{d}P_0$ denote Euclidean volume and perimeter. We may also refer to the perimeter of Ω as the perimeter of its boundary.

Definition 2 (Isoperimetric Region)**.** A region $\Omega \subset M$ is *isoperimetric* if it has the smallest weighted perimeter of all regions with the same weighted volume. The boundary of an isoperimetric region is also called isoperimetric.

We can generalize the idea of an isoperimetric region by considering two or more volumes.

Definition 3 (Isoperimetric Cluster)**.** An isoperimetric cluster is a set of disjoint open regions Ω_i of volume V_i such that the perimeter of the union of the boundaries is minimized.

For example, in the plane with density 1, optimal clusters are known for one area, two areas (Foisy et al. [6]), three areas (Wichiramala [14]), and four equal areas (Paolini and Tortorelli [10]), as in Figure 1. Note that for density r^p, scalings of minimizers are minimizers, because scaling up by a factor of λ scales perimeter by λ^{p+1} and area by λ^{p+2}.

The following proposition summarizes existence, boundedness and regularity of isoperimetric clusters in $\mathbb{R}^n$ with density, proved by Hirsch et al. [7] following such results for single bubbles (Rosales et al. [12, Thm. 2.5]) and Morgan and Pratelli [9, Thm. 5.9]).

Proposition 4 (Existence, Boundedness and Regularity)**.** *Consider $\mathbb{R}^n$ with radial nondecreasing C^1 density f such that $f(r) \to \infty$ as $r \to \infty$. An isoperimetric cluster that encloses and separates given volumes exists (Hirsch et al. [7, Thm. 2.5]) and is bounded (Hirsch et al. [7, Prop. 2.6]). In $\mathbb{R}^2$, the cluster consists of smooth curves with constant generalized curvature meeting in threes at $120°$ except possibly at points where the density vanishes (Hirsch et al. [7, Thm. 2.8]). The C^1 hypothesis may be allowed to fail, for instance at isolated points.*

Note that, unlike Hirsch et al. [7], by definition we allow a density to vanish at isolated points.

Definition 5 (Generalized Curvature)**.** In R^2 with density f, the generalized curvature κ_f of a curve with inward-pointing unit normal N is given by the formula

$$\kappa_f = \kappa_0 - \frac{\partial \log f}{\partial N},$$

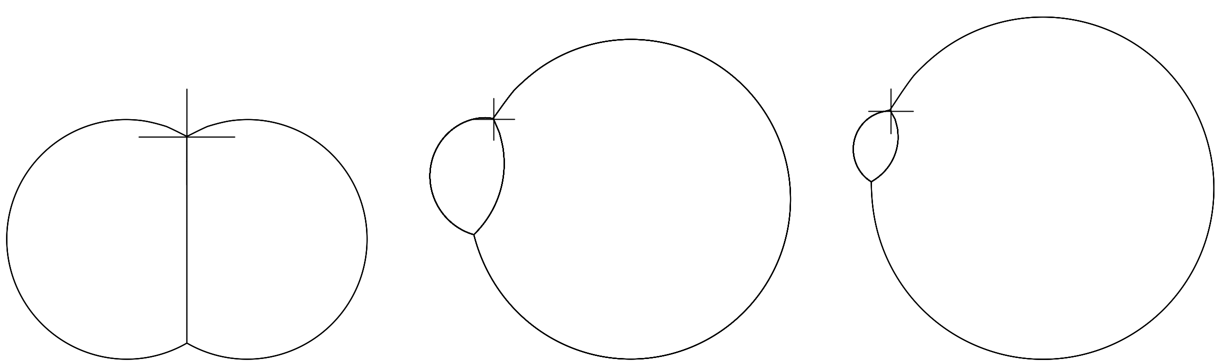
where κ_0 is the (unweighted) geodesic curvature. This comes from the first variation formula, so that generalized curvature has the interpretation as minus the perimeter cost $\mathrm{d}P/\mathrm{d}A$ of moving area across the curve, and constant generalized mean curvature is the equilibrium condition $\mathrm{d}P = 0$ if $\mathrm{d}A = 0$ (see [12, Sect. 3]).

Remark 6. In R^2 with density r^p, a circular arc has constant generalized curvature if and only if the circle passes through the origin [7, Rem. 2.9].

3 Multiple Bubbles in R^2 with density r^p

With Brakke's Evolver [2], Proposition 7 supports the conjecture of Hirsch et al. [7] that the optimal double bubble in the plane with density r^p is the standard double bubble. Proposition 8 provides a conjecture on the form of the triple bubble, and Proposition 10 provides a conjecture on the form of the quadruple bubble.

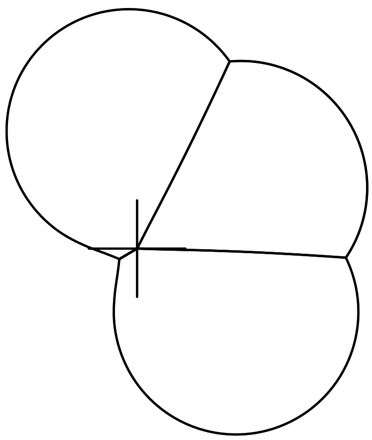
Proposition 7 (Double Bubble)**.** *Computations on Brakke's Evolver [2] support the conjecture [7] that the optimal planar double bubble for density r^p consists of a standard double bubble with one vertex at the origin, as shown in Figure 2.*



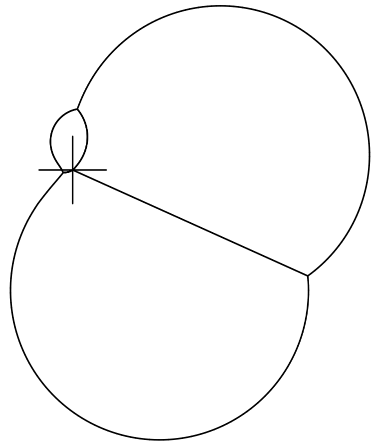
(a) Equal areas (b) Areas 10 and 1000 (c) Areas 1 and 1000

Figure 2: Computations in Brakke's Evolver [2] in $\mathbb{R}^2$ with density r^2 support the conjecture that the optimal double bubble is the standard double bubble with one vertex at the origin (marked here by a plus). Densities r^5, r^3 and $r^{0.5}$ are apparently identical.

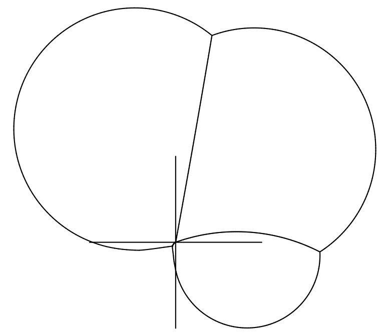
Proposition 8 (Triple Bubble)**.** *Computations with Brakke's Evolver [2] indicate that the optimal triple bubble in the plane with density r^p consists of three circular arcs meeting at the origin, one shrinking as p increases, separating the bubbles from each other, and three constant-generalized-curvature curves (see Remark 6), separating the bubbles from the exterior, as in Figure 3 and Figure 4.*



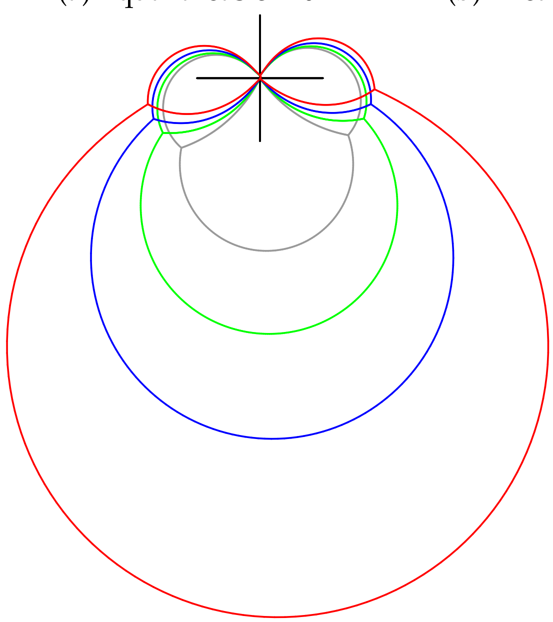
(a) Equal areas of 10



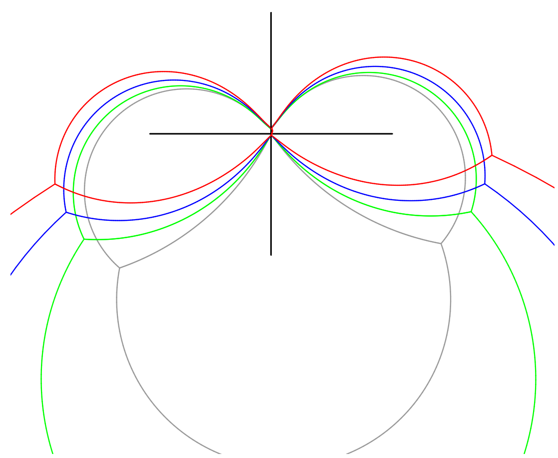
(b) Areas 0.1, 100 and 100



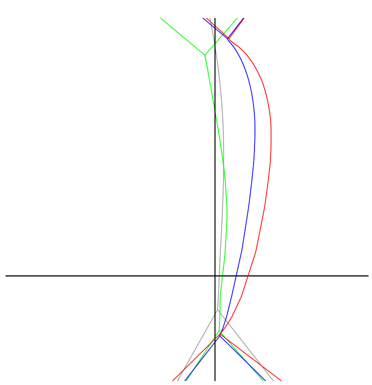
(c) Areas 1, 1 and 0.1



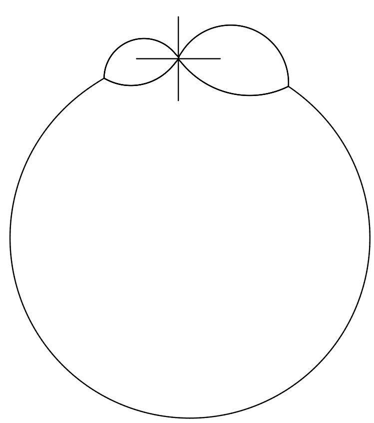
(d) Areas 0.1, 0.1 and 1 (grey), 0.1, 0.1 and 5 (green), 0.1, 0.1 and 20 (blue) and 0.1, 0.1 and 100 (red) overlayed



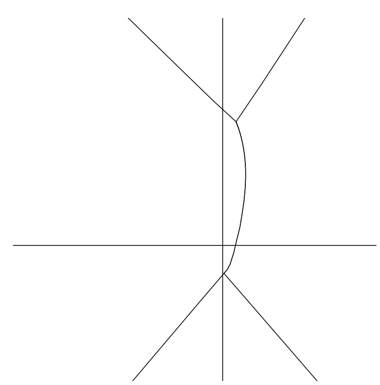
(e) Overlay zoomed in on the origin



(f) Overlay zoomed in further reveals small edges

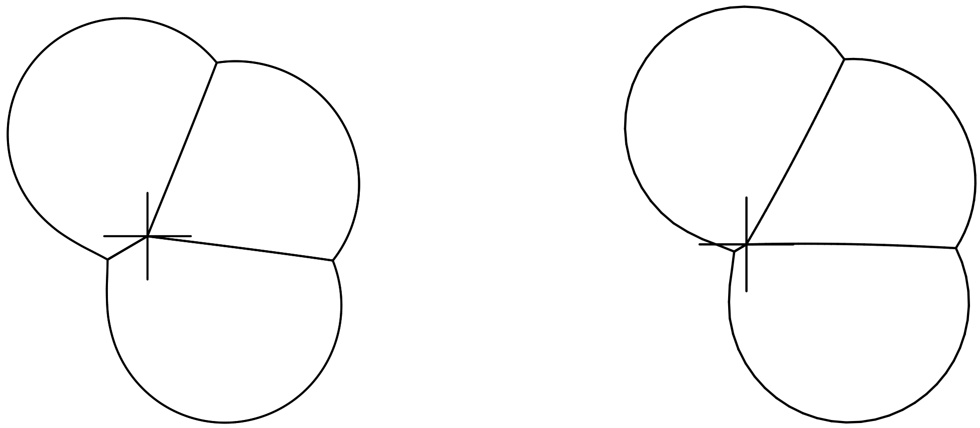


(g) Volumes 0.1, 0.5 and 100



(h) Zoomed in on the origin

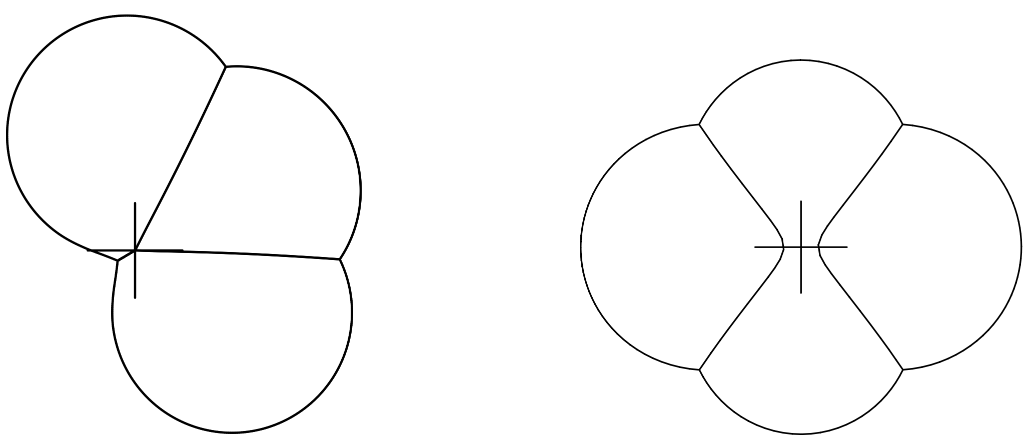
Figure 3: Computations in Brakke's Evolver in $\mathbb{R}^2$ with density r^2 suggest that the optimal planar triple bubble consists of three circular arcs meeting at the origin and three nearly circular arcs meeting near the origin. Areas are labelled clockwise starting at the upper left bubble. Densities r^3, r^4, r^5, r^6 and r^7 are apparently similar.



(a) A triple bubble with equal areas for $p = 1.7$ (b) A triple bubble with equal areas for $p = 1.9$

Figure 4: As p increases from 0, one edge gets shorter, moving a second vertex near the origin.

Proposition 9. *Our conjectured triple bubble of Proposition 8 has less perimeter than three bubbles in a linear chain, as in Figure 5. The linear chain evolves toward our conjectured triple bubble as in Figure 6.*



(a) Our conjectured triple bubble with equal volumes of 10 has perimeter just over 63 (b) A linear chain with equal volumes of 10 has perimeter just over 66

Figure 5: Our conjectured triple bubble has less perimeter than a linear chain in the plane with density r^2. Densities $r^{0.5}$ and r^3 are apparently similar.

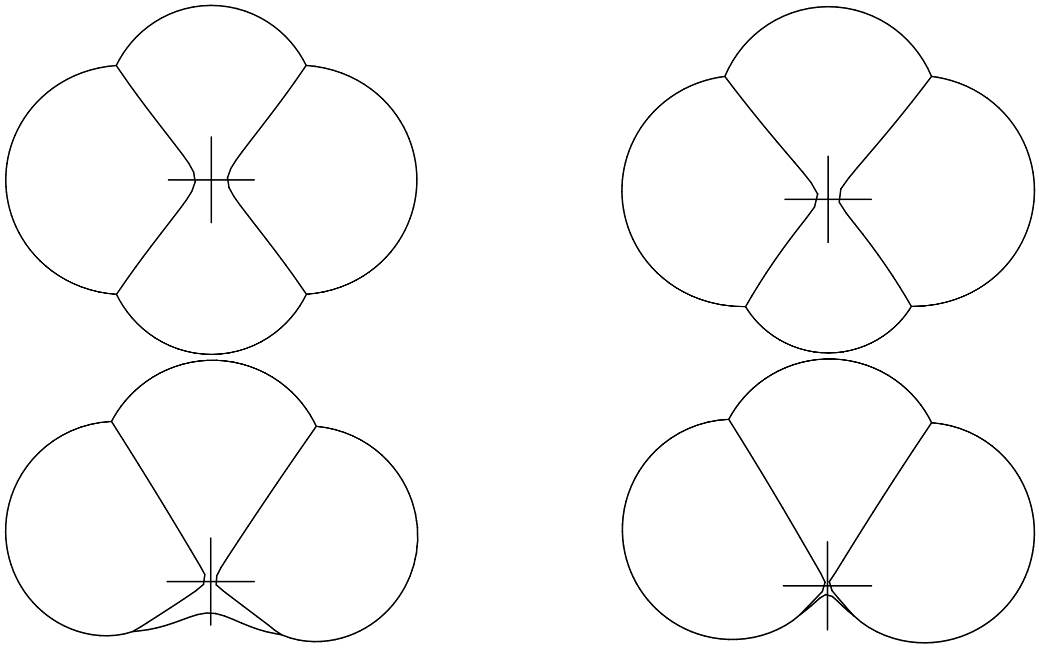
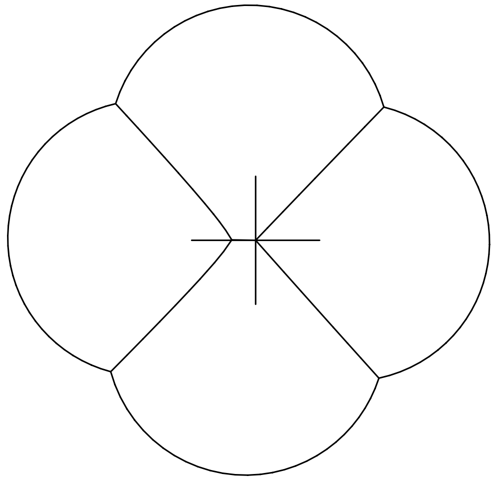


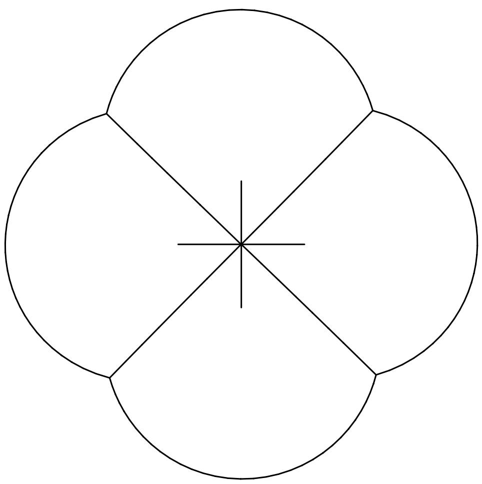
Figure 6: A linear chain, after slight displacement upwards, evolves as far as possible (without changing its topological type) toward our conjectured triple bubble, here in the plane with density r^2. Densities r^3 and r^4 are apparently identical.

Proposition 10 (Quadruple Bubble)**.** *Computations with Brakke's Evolver [2] indicate that, for the optimal quadruple bubble in the plane with density* r^p*, as* p *increases from* 0 *(the standard Euclidean quadruple bubble as in Figure 1), the central edge with one endpoint at the origin shrinks as in Figure 7 until it disappears when* p *reaches* 1*, after which four circular arcs meet at the origin (where the density vanishes) as in Figure 8.*

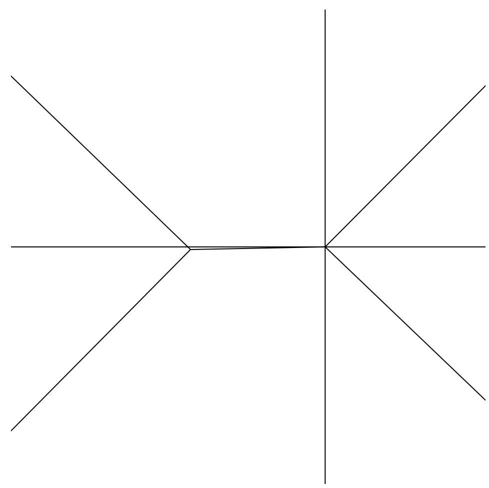
For example, in the plane with density r^p for $p \geq 1$ our conjectured quadruple bubble of Proposition 10 has less perimeter than the Euclidean quadruple bubble, as in Figure 9.



(a) A quadruple bubble with equal areas for $p = 0.3$

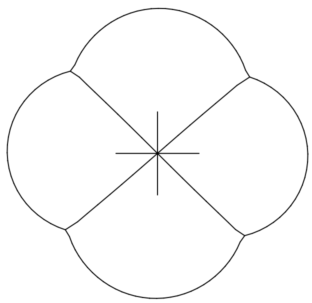


(b) A quadruple bubble with equal areas for $p = 0.99$

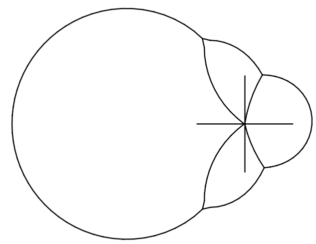


(c) The same quadruple bubble as Figure 7b zoomed in on the origin reveals a small edge

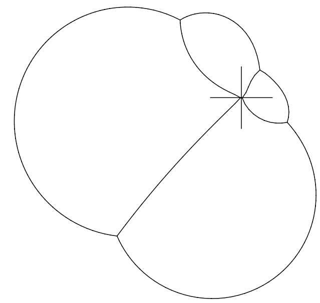
Figure 7: Computations in Brakke's Evolver [2] in $\mathbb{R}^2$ with density r^p for $p < 1$ suggest that the optimal planar quadruple bubble has a short central edge with one endpoint at the origin, shrinking as p increases toward 1.



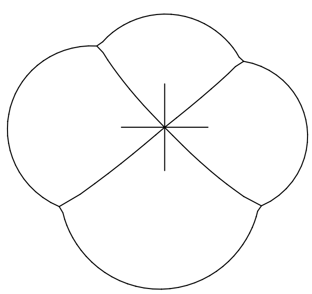
(a) Equal areas of 3 show a symmetry absent in the Euclidean case (1)



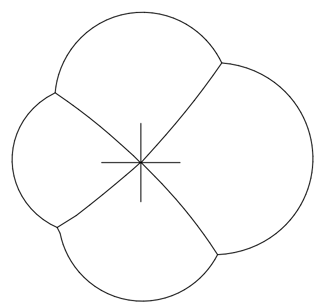
(b) Areas of 0.1, 0.1, 0.1 and 10



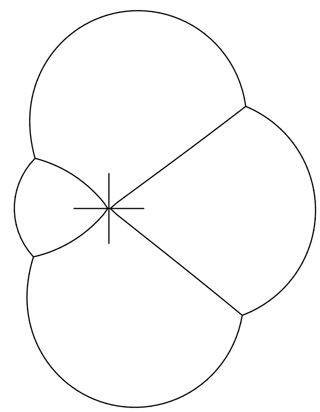
(c) Areas of 1, 0.1, 30 and 50



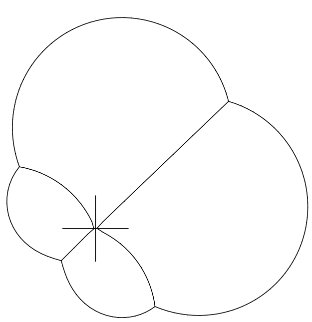
(d) Areas of 1, 2, 4 and 3



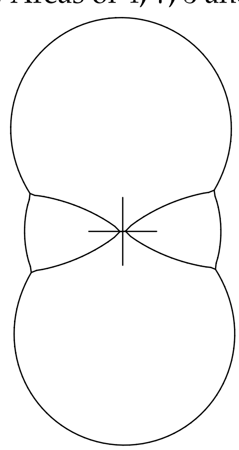
(e) Areas of 4, 7, 3 and 2



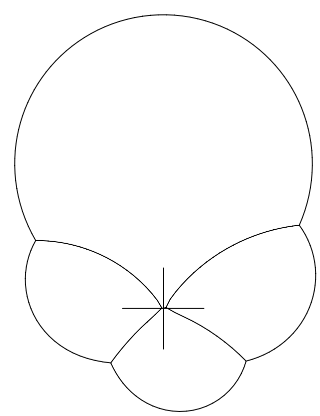
(f) Areas of 20, 20, 20 and 1



(g) Areas of 30, 30, 1 and 1, with total perimeter just under 104

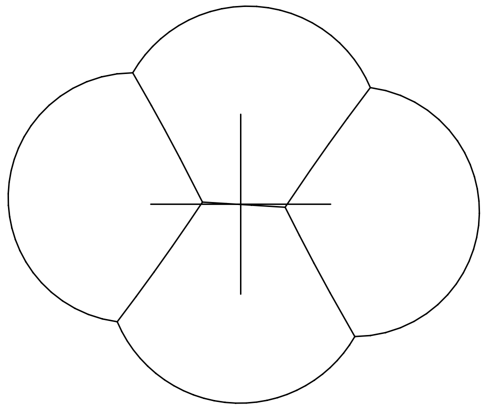


(h) An inferior version of Figure 8g with areas of 30, 1, 30 and 1, with total perimeter just over 106

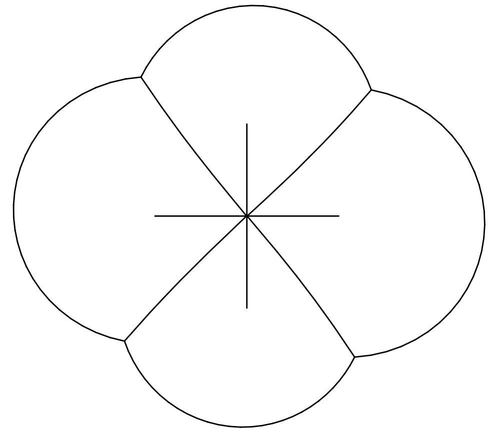


(i) Areas of 50, 3, 1 and 2

Figure 8: For $p \geq 1$, the central edge has collapsed and four circular arcs meet at the origin. The areas are labelled clockwise, starting from the top bubble. Densities r^3 and r^4 are apparently identical.



(a) The standard Euclidean quadruple bubble with unit Euclidean areas, centered at the origin in the plane with density r^2, has areas around 0.5, 0.8, 0.5 and 0.8 and perimeter around 10.86.

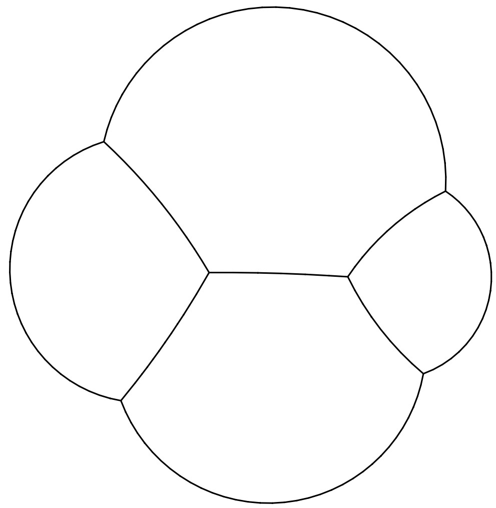


(b) Our conjectured quadruple bubble with the same areas in the plane with density r^2 has perimeter around 10.81.

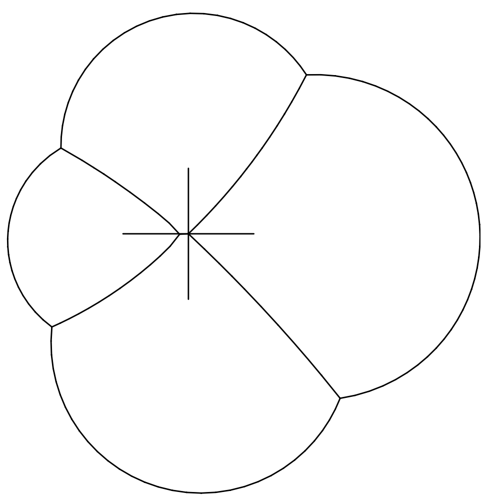
Figure 9: With the same areas in the plane with density r^2, our conjectured quadruple bubble has less perimeter than the standard Euclidean quadruple bubble.

Proposition 11. *Computations indicate that the optimal Euclidean quadruple bubble has the two largest areas on opposite sides of the central edge. As p increases towards 1, it prefers the largest and smallest bubbles on opposite ends of the central edge. Once the central edge has disappeared, for $p \geq 1$, the largest and smallest bubbles remain opposite. See Figure 10.*

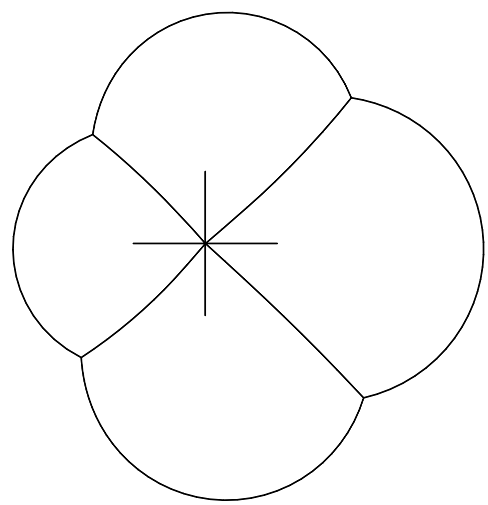
This arrangement of unequal areas is apparently a new conjecture even for the Euclidean case. It indicates that that the Euclidean quadruple bubble never has convex regions, so the results of [5] never apply. It is known to be true for small deformations of the equal-areas minimizer because it saves perimeter to shrink convex regions and expand nonconvex regions.



(a) The optimal Euclidean quadruple bubble apparently has the two largest regions on opposite sides of the central edge.



(b) As p increases towards 1, the optimal quadruple bubble has the largest and smallest regions on opposite ends of the central edge.

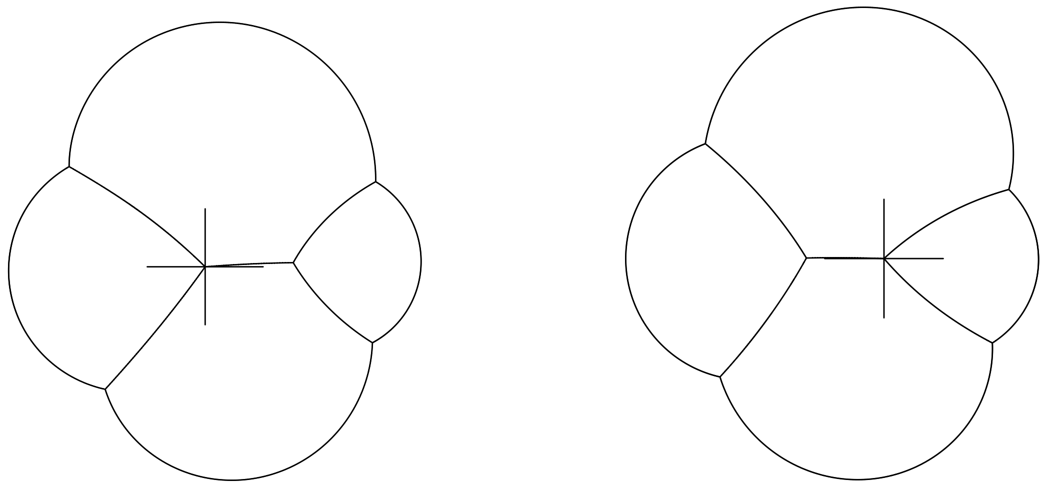


(c) Once the central edge has disappeared, for $p \geq 1$, the largest and smallest bubbles remain opposite.

Figure 10: Optimal quadruple bubble orderings for different values of p.

Proposition 12. *Computations indicate that when a central edge is present, our conjectured quadruple bubble of Proposition 10 is most effective with the vertex of the larger bubbles on the origin, as in Figure 11.*

This is the opposite of the $\mathbb{R}^1$ case where the vertex of the smallest bubbles is on the origin (Ross [13, Thm. 1]). In $\mathbb{R}^2$, larger bubbles require more perimeter, so it makes sense to have them closer to the origin.



(a) Our conjectured quadruple bubble with the vertex of the larger bubbles on the origin has perimeter around 17.67 .

(b) The quadruple bubble with the vertex of the smallest bubble on the origin has perimeter around 17.70 .

Figure 11: Our conjectured quadruple bubble has the vertex of the larger bubbles on the origin (here for density $r^{0.1}$).

Acknowledgements

This paper was written under the guidance of Frank Morgan remotely, following separate work done at the MathPath summer camp. The author would like to thank Professor Morgan for his advice and support. Additionally, the author would like to thank Professor Ken Brakke for generous help with his Evolver.

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