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Primeless and single-prime intervals

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1 Introduction

For millennia, mathematicians have been obsessed with prime numbers. They play an integral part in modern number theory, and are often involved in many mathematical proofs. In this paper, I explore primeless and single-prime intervals, and show clearly how to construct them. A decent understanding of factorials and elementary number theory is an important prerequisite before reading my paper.

2 A short proof of Euclid's theorem

Let's begin with a simple proof of the fact, first proved by Euclid around 300 BCE, that there are infinitely many prime numbers; that is, that there is no "largest prime". Assume that this fact is not true; that is, that there are only finitely many prime numbers. Let p be the largest of these and define the number $q = p!+1 = p(p-1)\cdots 2\cdot 1+1$. Since q is not divisible by any of the numbers $2, 3, \ldots, p$, its prime factors must be greater than p, a contradiction since p is the largest prime. Therefore, our assumption must be false, and so there are infinitely many primes. This simple proof is helpful in explaining the rest of the paper.

3 A word on the Prime Number Theorem

Proven by Hadamard and de la Vallee Poussin in 1896, the Prime Number Theorem helps us to understand the distribution of primes among the positive integers. One fact that we learn from this theorem is that, among the first N positive integers, the average gap between consecutive primes is approximately $\log(N)$. While proving the theorem is not the purpose of this paper, one intuition that can be derived from it is important: gaps between primes increase on average for larger primes.

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4 Primeless intervals

We now define primeless intervals:

> An interval $[a, b]$ of integers is *primeless* if it contains no prime number.

Example 1.
$[8, 10]$ and $[14, 16]$ are primeless intervals each of size 3.
$[25, 28]$ is a primeless interval of size 4.

We seek to construct a primeless interval of size n for each $n \geq 0$. While not apparent at first, a quick look at a property of factorials reveals a pretty straightforward method to construct such an interval. Consider the interval $I = [a, a + n - 1]$ for the integer

$$a = (n + 2)! + 2 = 1 \cdot 2 \cdots (n + 2) + 2$$

and note each integer in I has the form $a + m$ for some m with $0 \leq m < n$. Then

$$\begin{aligned} a + m &= 1 \cdot 2 \cdots (m + 1)(m + 2)(m + 3) \cdots (n + 2) + m + 2 \\ &= (m + 2)\big(1 \cdot 2 \cdots (m + 1)(m + 3) \cdots (n + 2) + 1\big) . \end{aligned}$$

Since $m + 2$ and $1 \cdot 2 \cdots (m + 1)(m + 3) \cdots (n + 2) + 1$ are both integers greater than 1, we have that $a + m$ is composite. We have therefore proved that every integer in the interval I is composite. We have then completed our construction and verified that

> For each $n \geq 0$, $\big[(n+2)! + 2, (n+2)! + n+1\big]$ is a primeless interval of size n.

5 Single-prime intervals

We now define single-prime intervals:

> An interval $[a, b]$ of integers is *single-prime* if it contains exactly 1 prime.

Example 2.
$[4, 6]$ and $[13, 15]$ are single-prime intervals each of size 3.
$[18, 21]$ is a single-prime interval of size 4.

We seek to construct a single-prime interval of size n for each $n \geq 0$. This is trickier than constructing a primeless interval, because we need exactly 1 prime and not at least 1 prime. However, instead of guessing random intervals of size n, we can adopt a more systematic approach.

Consider our primeless interval of size n given by $\big[(n+2)!+2, (n+2)!+n+1\big]$. Note that there are primes smaller than $(n+2)!+2$ as well as primes larger than $(n+2)!+n+1$ by Euclid's Theorem. Therefore, we can shift the primeless interval by -1 until the interval contains a prime. This would also work if we were to shift the primeless interval by $+1$, though it may take longer since gaps between primes frow on average for larger primes by the Prime Number Theorem. Either way, this simple algorithm guarantees that the new interval has exactly 1 prime.

6 Application

Suppose we want to construct a primeless interval and single-prime interval of size 8. Let's apply the constructions above.

6.1 A primeless interval of size 8

We directly apply the formula earlier in the paper with $n = 8$ to obtain $[10! + 2, 10! + 9]$ as the required interval. The simple illustration below explains why.

$$\begin{aligned}
10! + 2 &= 1 \cdot 2 \cdots 10 + 2 &= 2(1 \cdot 3 \cdots 10 + 1) \\
10! + 3 &= 1 \cdot 2 \cdots 10 + 3 &= 3(1 \cdot 2 \cdot 4 \cdots 10 + 1) \\
&\vdots \\
10! + 9 &= 1 \cdot 2 \cdots 10 + 9 &= 9(1 \cdot 2 \cdots 8 \cdot 10 + 1)
\end{aligned}$$

It's obvious that each integer in our interval is composite.

6.2 A single-prime interval of size 8

Let's shift our constructed interval $[10! + 2, 10! + 9]$ by $+1$ twice to obtain the interval $[10! + 4, 10! + 11]$. We have removed the composite numbers $10! + 2$ and $10! + 3$ from our interval and introduced two new integers $10! + 10$ and $10! + 11$, the last of which is prime. The interval $[10! + 4, 10! + 11]$ is therefore single-prime of size 8, as required.

7 Conclusion

I've always been fascinated by primes, and there's still so much about them to explore. The construction of an interval with a specified number of primes has never been a straightforward problem to solve. However, I show that in the case of a primeless and single-prime interval, the construction is easier than expected.