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The length of the path of a ray in a square

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1 Introduction

This article is about an example of "mathematical billiards", a subject which deals with a particle moving freely through a bounded two-dimensional region, and following the physical principle of reflection: whenever the particle reaches the boundary of the region, it bounces back with the angle of reflection equal to the angle of incidence; see Figure 1.

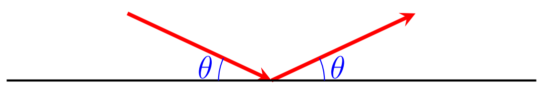

Figure 1: A particle reflecting off a line with the angle of incidence equal to the angle of reflection.

Such a system can be thought of as modelling the behaviour of a ray of light, or as an idealised version of the behaviour of a billiard ball, ignoring other physical forces on the ball like friction or air resistance. For a general introduction to this theory, see for example [1].

In this article, we consider a square region, with a ray of light emanating from a corner of the square. The angle of reflection is well defined when the ray hits a side of the square, but not when it hits a corner, so we will consider the path to be finished if/when the ray hits a corner.

The length of the ray's path is our main focus. The length of the path may be infinite, or the path may terminate at a corner. Which of these two scenarios occurs depends on the initial angles the ray makes with the sides of the square. In this study, we determine the condition for the ray's route to be finite, and compute its length in this case.

2 Method

Since this is a problem relating to the geometry of a square, it suffices to consider the unit square in the Cartesian plane: the set of points (x, y) satisfying $0 \leq x, y \leq 1$. Without loss of generality, we further assume that the ray emanates from the origin at an angle θ of less than or equal to $\frac{\pi}{4}$ with the x-axis; otherwise, the ray would make an angle of less than or equal to $\frac{\pi}{4}$ with the y-axis.

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2.1 The first trip up

If the angle θ is exactly $\frac{\pi}{4}$, then the ray simply goes to the point $(1, 1)$ and terminates. Otherwise, the ray will hit the right side of the square at the point $(1, \tan\theta)$ and reflect back towards the left. At this point, there are three possibilities. If $2\tan\theta$ is less than 1, then the ray will hit the left side of the square at the point $(0, 2\tan\theta)$ and then reflect back towards the right. On other hand, if $2\tan\theta$ is larger than 1, then the ray will hit the top of the square before reaching the left hand side and reflect downwards. Finally, if $2\tan\theta = 1$, then the ray will hit the upper left corner of the square and terminate; see Figure 2.

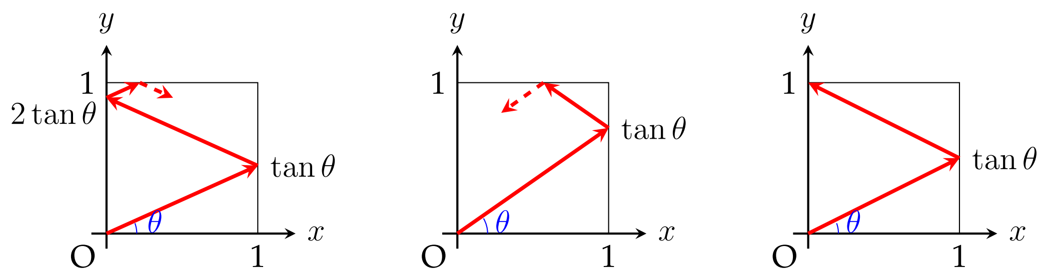


Figure 2: In the first picture, $\tan\theta = .45$, and the ray hits each side of the square once before reaching the top and reflecting back down. In the second picture, $\tan\theta = 0.7$ and the ray hits the right side once before hitting the top and reflecting back down. In the third picture, $\tan\theta = 0.5$ and the ray hits the right side before hitting the upper left corner and terminating.

More generally, the ray will hit the sides of the square n times on the way up, where n is the largest integer such that $n\tan\theta \leq 1$. We can express n using the floor function as $n = \lfloor\frac{1}{\tan\theta}\rfloor = \lfloor\cot\theta\rfloor$, where $\lfloor x\rfloor$ means the greatest integer that is less than or equal to x. We will also want to use the ceiling function $\lceil x\rceil$, which means the smallest integer that is greater than or equal to x.

We can think of these $\lfloor\cot\theta\rfloor$ first legs of the path as each being the hypotenuses of a right triangles with base 1 and height $\tan\theta$. We call such triangles *unit right triangles*. The last leg of the triangle, assuming the ray does not terminate at one of the top corners (which happens precisely when $\cot(\theta)$ is an integer) will be the hypotenuse of a similar right triangle, but with height $1 - \lfloor\cot\theta\rfloor\tan\theta$. We can compute the base of this right triangle using its similarity with the unit right triangles as

$$\frac{1 - \lfloor\cot\theta\rfloor\tan\theta}{\tan\theta} = \cot\theta - \lfloor\cot\theta\rfloor .$$

We call such a triangle a *proportional right triangle*; see Figure 3.

Suppose that $\cot\theta$ is not an integer, so that the ray hits the top of the square not in a corner, and reflects back down. By the reflection principle, this will again be the hypotenuse of a proportional right triangle, this time with base

$$1 - \big(\cot\theta - \lfloor\cot\theta\rfloor\big) = \lceil\cot\theta\rceil - \cot\theta .$$

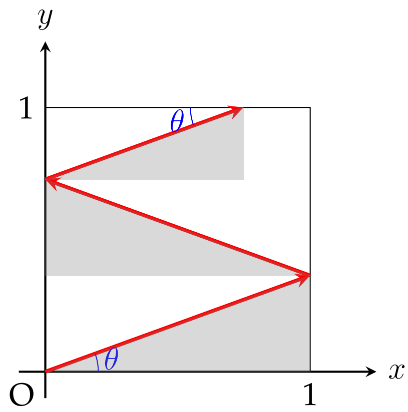


Figure 3: In this picture, the ray hits the sides twice before hitting the top. Therefore, $\lfloor \cot\theta \rfloor = 2$, and the path determines two unit right triangles and a smaller proportional right triangle. The proportional right triangle has height $1 - 2\tan\theta$ and base $\cot\theta - 2$.

We can again compute the height of this right triangle using its similarity with a unit right triangle as

$$\big(\lceil \cot\theta \rceil - \cot\theta\big)\tan\theta = \lceil \cot\theta \rceil \tan\theta - 1\,.$$

A key observation here is that if we add the heights of the two proportional right triangles formed by the last leg before hitting the top of the square and from the first leg after reflecting back down, we get

$$\big(1 - \lfloor \cot\theta \rfloor \tan\theta\big) + \big(\lceil \cot\theta \rceil \tan\theta - 1\big) = \big(\lceil \cot\theta \rceil - \lfloor \cot\theta \rfloor\big)\tan\theta = \tan\theta\,.$$

which is the height of a unit right triangle. This should not be surprising, since these two triangles can be thought of as coming from a single unit right triangle “folded down” at the top of the square; see Figure 4.

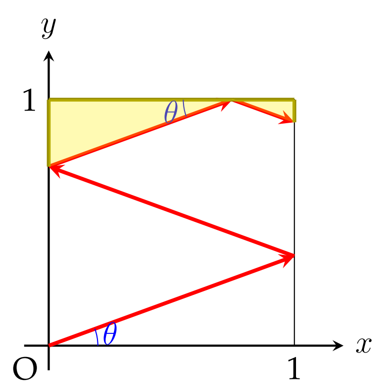


Figure 4: The yellow proportional right triangle on the upper left has height $1 - 2\tan\theta$ and base $2 - \cot\theta$. The yellow proportional right triangle on the upper right has base $\cot\theta - 1$ and height $3\tan\theta - 1$. The sum of the two triangles's heights is $\tan\theta$, which is also the height of a unit right triangle, and also their common ratio of height to base.

2.2 The path back down

We continue with our assumption that $\cot\theta$ is not an integer, so that the ray hits the top of the square not in a corner. The ray will then continue bouncing off the sides of the square on the way down, forming (hypotenuses of) unit right triangles with each leg, until it either reaches a bottom corner, or reaches a point on a side of the square which is less than $\tan\theta$ from the bottom. In the latter case, it will make a smaller proportional right triangle with the last leg, hit the bottom of the square, and make another proportional right triangle on the next leg back up. By the same argument as above, the heights of these two proportional right triangles will add up to $\tan\theta$, the height of a unit right triangle.

2.3 Conclusion

These observations imply the following theorem and corollary.

Theorem 1. *The ray hits the sides of the square each time the total vertical distance travelled increases by* $\tan\theta$.

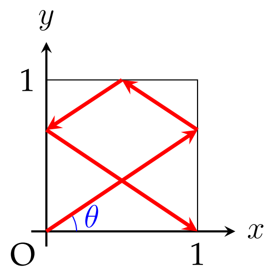
Proof. The preceding analysis shows that the ray hits the side of the square each time it forms the hypotenuse of a unit right triangle, which has height $\tan\theta$; and each time it forms hypotenuses of two proportional right triangles from reflecting off the top or bottom of the square, the sum of whose heights is again $\tan\theta$. □

Corollary 2. *The ray will reach a corner of the square if and only if* $\tan\theta$ *is a rational number. In this case, let* $\tan\theta = \frac{m}{n}$ *where* m *and* n *are relatively prime positive integers. Then the total length of the path of the ray is* $n\sec\theta$.

Proof. The ray will hit a corner if it hits a side of the square at a point when the total vertical distance travelled is a positive integer. By the above theorem, this will happen at the first positive integer which is a multiple of $\tan\theta$, which exists if and only if $\tan\theta$ is rational. Let $\tan\theta = \frac{m}{n}$ where m and n are relatively prime positive integers. Then the vertical distance travelled before the ray hits a corner is n. Since each leg of the path is the hypotenuse of a right triangle which makes an angle of θ with its base, the vertical distance travelled is $\cos\theta$ times the total distance travelled, so that the total distance is $\sec\theta$ times the vertical distance. □

3 Another perspective: unfolding

There is another interesting way of looking at this problem. Instead of imagining the ray reflecting off the sides of a square, imagine that the ray continues in a straight line making an angle of θ with the x-axis, but that we "unfold" a copy of the square in the direction of travel of the ray at each point that the ray reaches a side of the square; see Figure 5.



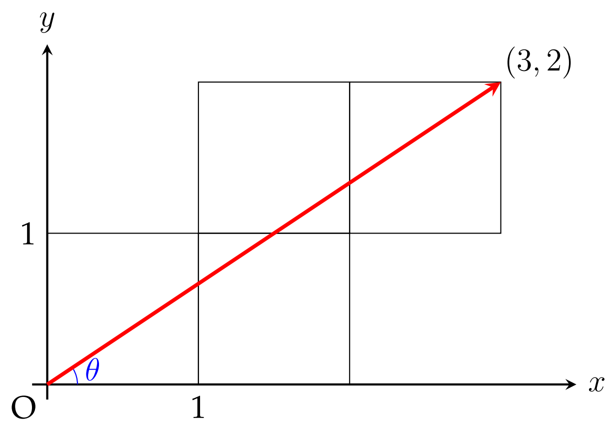


Figure 5: On the left is a ray in the square, with $\tan\theta = \frac{2}{3}$. There are four segments to the path: first, the ray hits the right side, then the top, then the left side, then terminates in the lower right corner. On the right is the same path "unfolded" in the plane. The four squares correspond to the four path segments, and the termination point $(3, 2)$ corresponds to the lower left corner of the original square.

From this perspective, we can see that the condition that the ray eventually reaches a corner in the original setup is equivalent to the condition that the ray with slope $\tan\theta$ emanating from the origin in the plane eventually crosses a point with integer coordinates. This happens precisely if $\tan\theta$ is rational, and the first such point crossed is (n, m), where $\frac{m}{n}$ is the reduced form of $\tan\theta$. The length of the path in the original setup is just the length of the vector (n, m) in $\mathbb{R}^2$, which $\sqrt{n^2 + m^2}$. This agrees with our previous answer: if $\tan\theta = \frac{m}{n}$, then $n\sec\theta = n\frac{\sqrt{n^2+m^2}}{n} = \sqrt{n^2 + m^2}$; see Figure 6.

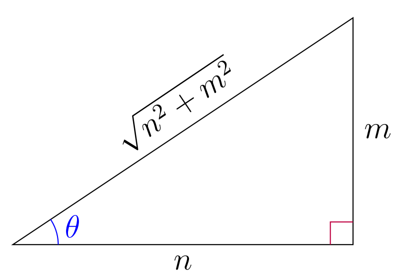


Figure 6: A right triangle with base n and height m, with the hypotenuse making an angle of θ with the base. We have $\sec\theta = \frac{\sqrt{n^2+m^2}}{n}$.

More generally, we can also consider the path of a ray in a rectangle whose ratio of height to base is rational. Without loss of generality, suppose the base of the rectangle is 1 and the height is a positive rational number a, which we consider in the Cartesian plane as the set of points (x, y) satisfying $0 \leq x \leq 1,\ 0 \leq y \leq a$. As before, we assume the ray emanates from the origin making an acute angle θ with the positive x-axis. Then a similar unfolding argument shows that the unfolded path will terminate at the first point (x, y) such that x is an integer and y is an integer multiple of a. Since for any point on the path we have $\frac{y}{x} = \tan\theta$, this condition is equivalent to x and $\frac{x\tan\theta}{a}$ both being integers. In particular, such an x exists if and only if $\tan\theta$ is rational, in which case the smallest such x is the denominator of the rational number $\frac{\tan\theta}{a}$ when expressed as a ratio of two relatively prime positive integers. The length of the path will then be

$$\sqrt{x^2 + y^2} = x \sec\theta\,.$$

For more on unfolding, see [1].

4 Conclusion

In this study, we have seen that the length of the path of a ray in a square which is subject to the reflection principle will be finite if and only if the initial angle it makes with a side of the square has rational tangent. In this case, we can compute the length of the path in terms of this angle.

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References

[1] S. Tabachnikov, *Geometry and Billiards*, American Mathematical Society, 2005.