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# Back to Babylonian roots

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## 1 Introduction to Babylonian maths

Nestled on a fertile plain between the banks of the Euphrates and Tigris rivers, Babylon emerges as a radiant jewel in the tapestry of ancient Mesopotamia[2]. Our journey begins in the 2nd millennium BCE, where the city's roots intertwine with the dawn of written civilisation.

As the centuries unfold, Babylon ascends from a provincial town to a majestic city-state, ruled by influential leaders like Hammurabi, whose Code stands as a testament to early legal systems. The Hanging Gardens[3], a verdant marvel, bloom as a symbol of artistic and engineering prowess. The city's architectural excellence reaches its zenith under the reign of Nebuchadnezzar II. The Ishtar Gate[4], adorned with mythical creatures, stands as a portal to Babylon's opulence. Nebuchadnezzar's ambitious projects, including the grand ziggurat Etemenanki[5], reflect the city's desire to touch the heavens. Babylon extends its influence across the region, forming the Babylonian Empire. The city becomes a melting pot of cultures, fostering trade, intellectual exchange and the flourishing of the arts and sciences.

Amidst the splendour of Babylon, a beacon of intellectual brilliance emerged, as the city-state delved into the realms of science and mathematics. The mathematicians used $9$ digits and calculated using a sexagesimal system, a system that was already used by ancient Sumerians in the 3rd millennium BC. Sexagesimal means that the numerical base is $60$, so that all positive integers are expressed as sums of powers of $60$ multiplied by coefficients which are positive integers less than $60$.[6] We can think of the calculations in sexagesimal as calculating time with $60$ seconds in a minute and $60$ minutes in the hour. That is the reason why I will adopt the presentation of the type $n_{60} = a''b'c^\circ$ for a number $n$ in base $60$ for its equivalent $n_{10} = a60^2 + b60 + c$ in base 10.

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[2]Mesopotamia is a word that comes from the greek word "mesos", which means "in the middle", and "potamos" which means "rivers". It is in modern-day Iraq.

[3]One of the Seven Wonders of the Ancient World.

[4]Ishtar is the goddess of love, war, and fertility.

[5]The name means "temple of the foundation of heaven and earth". It was a ziggurat dedicated to the Mesopotamian god Marduk in the ancient city of Babylon. It now exists only in ruins, located about 90 kilometres (56 mi) south of Baghdad, Iraq. Many scholars have identified Etemenanki as a likely inspiration for the biblical story of the Tower of Babel.

[6]We still use this system - in a modified form - for measuring time, angles and geographic coordinates.

This system is ideal for fractions, and for keeping accounts of harvests, since the number $60$ is a superior highly composite number with twelve factors[7] of which $2$, $3$ and $5$ are prime numbers. Furthermore, it is the least common multiple of $1$, $2$, $3$, $4$, $5$ and $6$.

There is recorded evidence that, in 1800 BCE, the Babylonian mathematicians could calculate Pythagorean triples – more than a thousand years before Pythagoras. The oldest known record comes from the Babylonian clay tablet Plimpton 322 [2]. This means that the Babylonians had expertise in multiplying and calculating squares of sexagesimal numbers.

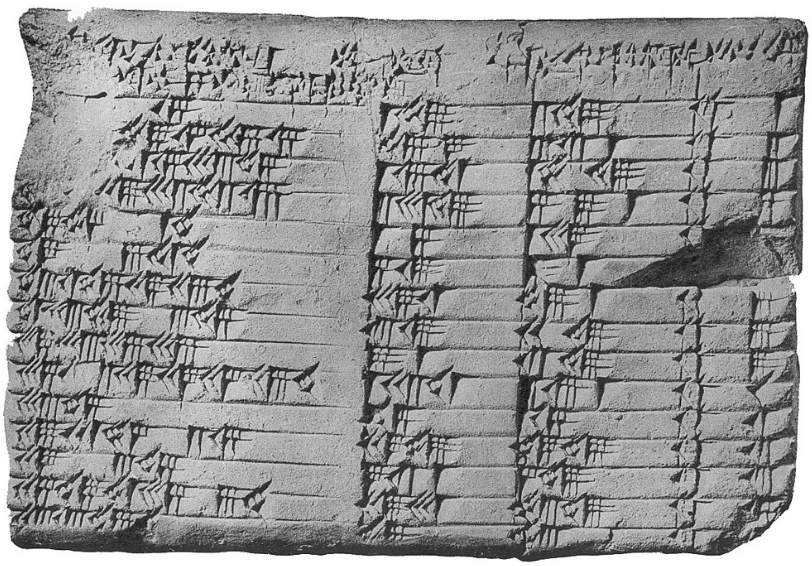

Figure 1: *Babylonian clay tablet showing Pythagorean triples. 1800 BCE*

Is it possible that this ability also allowed Babylonian mathematicians to calculate approximate values for square roots? Indeed, the YBC7289 clay tablet [1] illustrated in Figure 2[8], likely to be the work of a student between 1800BCE and 1600BCE, shows such an approximation:

$$\sqrt{2} \approx 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}\,.$$

There have been several conjectures about how this student obtained this result. Most conjectures are in favour of a geometric approach [3, 4] which could possibly be similar to the method used by Archimedes to obtain the approximation of $\pi$ in his treatise "Dimension of the Circle" (ca. 250 BCE), or similar to the method of Hero of Alexandria (1st century CE) [5]. Others have suggested techniques of division and averaging [6]. I believe this geometric approach derives from an influence of our Greek architectural heritage.

[7] Namely $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$.

[8] At Yale, the Institute for the Preservation of Cultural Heritage has produced a digital model of the tablet, suitable for 3D printing; see [7].

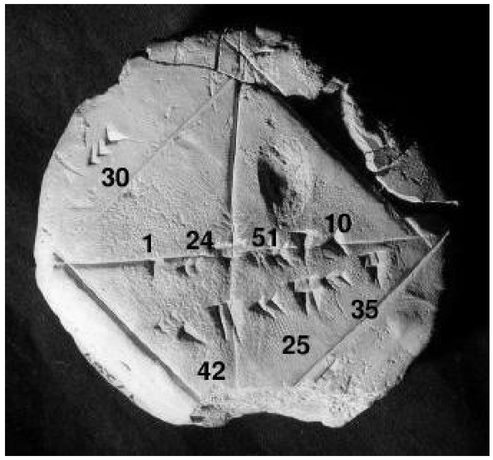


Figure 2: *Babylonian clay tablet showing the approximation of* $\sqrt{2}$*. 1800 BCE*

In this paper, I intend to argue in favour of a simple and efficient calculation. Indeed, the ability to calculate squares in the sexagesimal system seems very sufficient to quickly get the expected result. Though of course there is no evidence for how the calculation was done, and so this paper must be considered only as a credible conjecture.

We will attempt to multiply like a Babylonian student using an anachronistic multiplication array. Then, in three easy steps, we will converge towards the beautiful sexagesimal approximation.

However, I feel that it is important to first write down the following disclaimer. When studying mathematics from Antiquity, we should be careful to:

> Avoid reading into early mathematics ideas which we can see clearly today yet which may never have been in the mind of the authors.
>
> Never underestimate the significance of the mathematics just because it was produced by mathematicians who thought very differently from today's mathematicians.
>
> Realise that a great quantity of the mathematical achievements of ancient times, even if a great quantity were recorded on clay tablets or papyrus, may well have been lost.

Figure 3: *Babylonian clay tablet showing precalculated squares. 1800 BCE*

## 2 How to multiply like a Babylonian

The Babylonians used precalculated tables to assist with arithmetic. For example, two tablets found at Senkerah on the Euphrates in 1854, dating from 2000 BC, give lists of the squares of numbers up to $59$; see Figure 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $0^2 \to 9^2$ | $0'0^\circ$ | $0'1^\circ$ | $0'4^\circ$ | $0'9^\circ$ | $0'16^\circ$ | $0'25^\circ$ | $0'36^\circ$ | $0'49^\circ$ | $1'4^\circ$ | $1'21^\circ$ |
| $10^2 \to 19^2$ | $1'40^\circ$ | $2'1^\circ$ | $2'24^\circ$ | $2'49^\circ$ | $3'16^\circ$ | $3'45^\circ$ | $4'16^\circ$ | $4'49^\circ$ | $5'24^\circ$ | $6'1^\circ$ |
| $20^2 \to 29^2$ | $6'40^\circ$ | $7'21^\circ$ | $8'4^\circ$ | $8'49^\circ$ | $9'36^\circ$ | $10'25^\circ$ | $11'16^\circ$ | $12'9^\circ$ | $13'4^\circ$ | $14'1^\circ$ |
| $30^2 \to 39^2$ | $15'0^\circ$ | $16'1^\circ$ | $17'4^\circ$ | $18'9^\circ$ | $19'16^\circ$ | $20'25^\circ$ | $21'36^\circ$ | $22'49^\circ$ | $24'4^\circ$ | $25'21^\circ$ |
| $40^2 \to 49^2$ | $26'40^\circ$ | $28'1^\circ$ | $29'24^\circ$ | $30'49^\circ$ | $32'16^\circ$ | $33'45^\circ$ | $35'16^\circ$ | $36'49^\circ$ | $38'24^\circ$ | $40'1^\circ$ |
| $50^2 \to 59^2$ | $41'40^\circ$ | $43'21^\circ$ | $45'4^\circ$ | $46'49^\circ$ | $48'36^\circ$ | $50'25^\circ$ | $52'16^\circ$ | $54'9^\circ$ | $56'4^\circ$ | $58'1^\circ$ |

As you can see in this table, the squares in the the columns are arranged in neat sequences, and the memorisation of it seems pretty accessible. With this simple tool the Babylonians easily multiplied numbers, without the need for a mind-boggling $60 \times 60$ multiplication table, using this simple formula:

$$ab = \frac{1}{2}\big(a^2 + b^2 - (a-b)^2\big) .$$

For example,

$$51^\circ \times 25^\circ = \frac{1}{2}\big((51^\circ)^2 + (25^\circ)^2 - (26^\circ)^2\big) = \frac{1}{2}\big(43'21^\circ + 10'25^\circ - 11'16^\circ\big) = \frac{42'30^\circ}{2} = 21'15^\circ.$$

# 3 Easy multiplication array

To assist with calculations in the following section, I will first describe in this section an algorithm to carry out large multiplications in base $60$[9]. It is a version of what is sometimes called *lattice multiplication* or *tableau multiplication*, and has appeared in various forms in many different cultures and civilizations, including China, India, the Arab world and Europe. See [8] for a brief survey.

The purpose of this method is first to increase speed and lower the risk of mistakes[10] and second to maintain the possibility of increasing the size of the multipliers while keeping the same presentation.

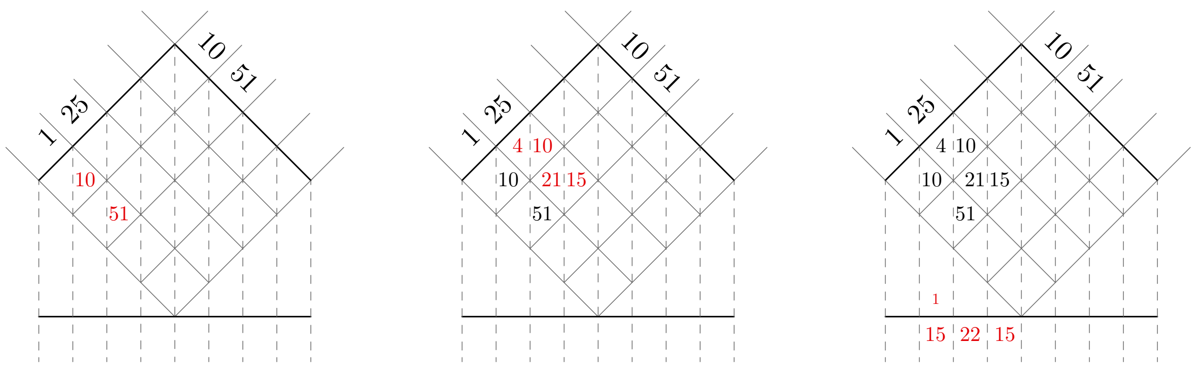


Figure 4: *Easy multiplication array*

For example, let's calculate

$$1'25° \times 10'51° .$$

The basic principle is to consider the multipliers as lengths and the result as a calculated area. Two multipliers composed respectively of $n$ and $m$ figures will generate a multiplication array of $n \times m$ interim multiplication results as illustrated in Figure 4. The interim results can be calculated in any order and there are no "carries" at this stage.

We complete the multiplication array in any order. When it is complete, we proceed to the additions of each column, in a traditional fashion, from right to left. When the sum of a column has the form $x60 + y$, we write down $y$ and we carry $x$ to the next column left.

In this example, the result of the multiplication is

$$1'25° \times 10'51° = 15''22'15° ,$$

or, in base $10$,

$$85 \times 651 = 55\,335 .$$

[9]The algorithm actually works in any base system, as illustrated in Figure 11.

[10]I had the chance to test it with 8-year-old kids - in base 10 - and it does increase speed and drastically reduces the risk of mistakes for large multiplications. I welcome feedback from school teachers.

Now suppose we choose to increase the multiplier $10'51^\circ$, for example multiplying it by $60$ and adding $41^\circ$. How do we proceed? We simply tuck $41^\circ$ on the end to get $10"51'41^\circ$. We then add its multiplication with very few simple steps, as illustrated in Figure 5.

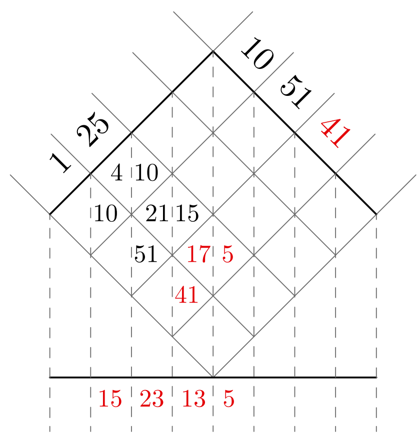


Figure 5: *Increasing one of the multipliers leads to a larger array, with the new pieces in red.*

The result of the multiplication is

$$1'25^\circ \times 10''51'41^\circ = 15'''23''13'5^\circ\,,$$

which is simple and lean in sexagesimal and a bit more impressive in decimal:

$$85 \times 39\,101 = 3\,323\,585\,.$$

## 4 Let's calculate an approximation of $\sqrt{2}$

In this section, we will try to approximate $\sqrt{2}$ using sexagesimal integer arithmetic. First, let's obtain a rough approximation. Note that for any square, the ratio of the length of a diagonal to the length of a side is equal to the square root of $2$. So we start by drawing a square of side $1'0^\circ$ (which is $60$ in base $10$) and measuring the length of the diagonal $d$. We find that $d$ is between $1'24^\circ$ and $1'25^\circ$ (respectively $84$ and $85$ in base $10$); see Figure 6. Therefore, the square root of $2$ is in between

$$\frac{84}{60} \quad \text{and} \quad \frac{85}{60}\,.$$

Now imagine that we could zoom in and divide our units into $60$ pieces each. Then, subject to our measuring capabilities, we could tighten the upper and lower bounds. We could then recursively iterate this process to obtain increasingly sharp approximations to the square root of $2$.

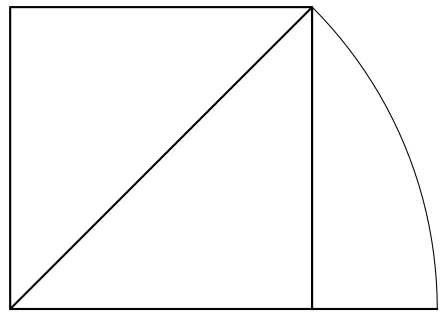


Figure 6: The diagonal of a square with side length $1'0^\circ = 60$ has length approximately $1'24^\circ = 84$.

I am not asking you to get out a magnifying glass just yet. Instead, the plan is to replicate this geometric process with an arithmetic tool that we know the Babylonians had at their disposal: the ability to multiply sexagesimal numbers.

We can verify by direct multiplication that

$$(1'24^\circ)^2 < 2''0'0^\circ < (1'25^\circ)^2 \,.$$

We will concentrate on the lower bound, which we may interpret, after dividing by $60^2$, as giving the approximation

$$\sqrt{2} \approx 1 + \frac{24}{60} \,.$$

To extend this approximation by another sexagesimal digit, we would like to find the largest integer $x$ such that

$$1 + \frac{24}{60} + \frac{x}{60^2} < \sqrt{2} \,.$$

We can express this inequality in terms of integer multiplication by writing

$$\left(1 + \frac{24}{60} + \frac{x}{60^2}\right)^2 < 2$$

and then multiplying by $60^4$ to get

$$\left(60^2 + 24 \cdot 60 + x\right)^2 < 2 \cdot 60^4 \,.$$

Expanding the square, we may further rewrite the inequality as

$$(60^2 + 24 \cdot 60)^2 + x^2 + 2x \cdot (60^2 + 24 \cdot 60) < 2 \cdot 60^4 \,,$$

or, in other words, as

$$x = \frac{2 \cdot 60^3 - (60+24)^2 \cdot 60}{2 \cdot (60+24)} - \frac{x^2}{2 \cdot 60 \cdot (60+24)} \,,$$

where we have cancelled a factor of $60$ in the first summand.

In sexagesimal notation, this inequality takes the form

$$x < \frac{2''' - (1'24^\circ)^2 \cdot 1'}{2^\circ \cdot 1'24^\circ} - \frac{x^2}{2' \cdot 1'24^\circ} = \frac{2''24'}{2'48^\circ} - \frac{x^2}{2''48'}\,,$$

where we have used the multiplication calculation $(1'24^\circ)^2 = 1''57'36^\circ$; see Figure 7.

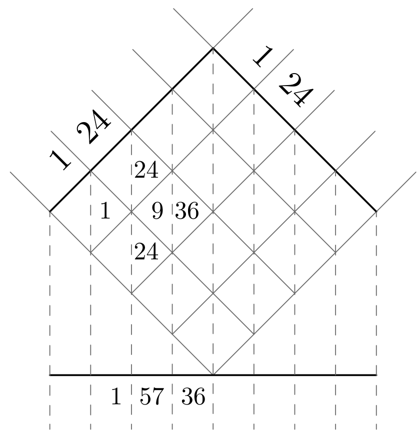


Figure 7: *Approximation of* $\sqrt{2}$ *- Step 1*

Although we are dealing with a quadratic inequality, we are searching for an integer $x$ between $0$ and $59$, and so the quadratic term is relatively small:

$$\frac{x^2}{2''48'} < \frac{60^2}{2 \cdot 60^2 + 48 \cdot 60} = \frac{5}{14}\,.$$

Therefore, if we simply ignore the quadratic term and take

$$x_0 = \left\lfloor \frac{2''24'}{2'48^\circ} \right\rfloor = \left\lfloor \frac{8640}{168} \right\rfloor = \left\lfloor \frac{360}{7} \right\rfloor = 51\,,$$

then we are guaranteed that $x_0$ is within short distance to one of the largest possible integers satisfying the original inequality. More precisely, either $x$ will satisfy the inequality, in which case $x$ is the largest such integer; or $x$ will not satisfy the inequality, in which case $x - 1$ must be the largest integer to satisfy the inequality. In particular, since the fractional part of $\frac{360}{7}$, which is $\frac{3}{7}$, is greater than the bound on the quadratic term of $\frac{5}{14}$, we can see that $x_0$ satisfies the inequality. Therefore,

$$1 + \frac{24}{60} + \frac{51}{60^2}$$

is the best lower approximation of $\sqrt{2}$ to two sexagesimal places, and we are done.

To add another sexagesimal digit to the approximation, we similarly search for the largest integer $y$ such that

$$\left(1 + \frac{24}{60} + \frac{51}{60^2} + \frac{y}{60^3}\right)^2 < 2\,,$$

which we again rearrange as

$$y < \frac{2 \cdot 60^5 - (60^2 + 24 \cdot 60 + 51)^2 \cdot 60}{2 \cdot (60^2 + 24 \cdot 60 + 51)} - \frac{y^2}{2 \cdot 60 \cdot (60^2 + 24 \cdot 60 + 51)}\,,$$

or in sexagesimal notation

$$y < \frac{2''''' - (1''24'51^\circ)^2 \cdot 1'}{2^\circ \cdot 1''24'51^\circ} - \frac{y^2}{2' \cdot 1''24'51^\circ}\,.$$

Again the quadratic term is very small:

$$\frac{y^2}{2' \cdot 1''24'51^\circ} < \frac{60^2}{2 \cdot 60 \cdot (60^2 + 24 \cdot 60 + 51)} = \frac{10}{1697} \approx 0.006\,.$$

So we concentrate on the first term and set

$$\begin{aligned} y_0 &= \left\lfloor \frac{2''''' - (1''24'51^\circ)^2 \cdot 1'}{2^\circ \cdot 1''24'51^\circ} \right\rfloor \\ &= \left\lfloor \frac{2''''' - 1''''59'''59''31''21'}{2''49'42^\circ} \right\rfloor = \left\lfloor \frac{28''39'}{2''49'42^\circ} \right\rfloor = \left\lfloor \frac{102540}{10182} \right\rfloor = 10\,, \end{aligned}$$

where we have used the multiplication calculation $(1''24'51^\circ)^2 = 1''''59'''59''31'21^\circ$; see Figure 8.

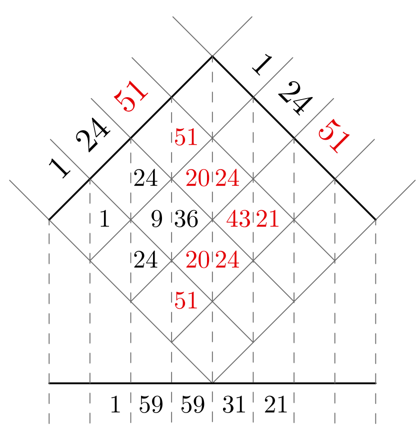


Figure 8: *Approximation of* $\sqrt{2}$ *- Step 2*

Again, the fractional part of $\frac{17090}{1697}$, which is $\frac{120}{1697}$, is greater than the bound of $\frac{10}{1697}$ on the quadratic term of the inequality, so $y_0$ satisfies the inequality. Therefore,

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$$

is the best lower approximation of $\sqrt{2}$ to three sexagesimal places.

Another way of looking at the method above is as follows. Consider a square with sides $u$ and area $u^2$, and a slightly larger square with sides $v = u + h$ and area $v^2 = (u+h)^2$. We have the first order approximation

$$v^2 = (u+h)^2 = u^2 + 2uh + h^2 \approx u^2 + 2uh\,,$$

assuming that $h$ is very small. Therefore, we can approximate the increment $h$ in the side of a square needed to increase its area from $u^2$ to $v^2$ by

$$h \approx \frac{v^2 - u^2}{2u}\,;$$

see Figure 9.

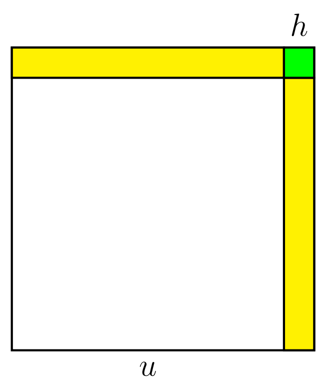


Figure 9: *A square of size $u$ is shown inside a square of size $v = u + h$. The difference in areas of the the two large squares is the sum of the areas of the three shaded rectangles: $v^2 - u^2 = 2uh + h^2$. If $h$ is small relative to $u$, then we may disregard the green square with the small area $h^2$ and approximate the difference in areas of the large squares as the sum of the areas of the yellow rectangles: $v^2 - u^2 \approx 2uh$.*

This is exactly the calculation we have done above (twice), with $v^2 = 2$ and $u$ a starting approximation for $\sqrt{2}$. After truncating $h$ appropriately using a floor function, we then end up with a better approximation for $\sqrt{2}$. For example, starting with $v^2 = 2$ and $u = 1 + \frac{24}{60}$, we have

$$h = v - u \approx \frac{v^2 - u^2}{2u} = \frac{1}{70}\,,$$

which is between $\frac{51}{60^2}$ and $\frac{52}{60^2}$; we truncated to $\frac{51}{3600}$ for a lower approximation.

The approximate ratio that we are using in these calculations for the difference in area $v^2 - u^2$ and the difference in side length $h$ is $2u$, which from the perspective of calculus is the first derivative of the squaring function evaluated at $u$. More precisely, we are approximating the difference $h$ in the square root function in terms of the difference of two squares $v^2 - u^2$, so we multiply $v^2 - u^2$ by $\frac{1}{2u}$, which is the first derivative of the square root function evaluated at $u^2$.

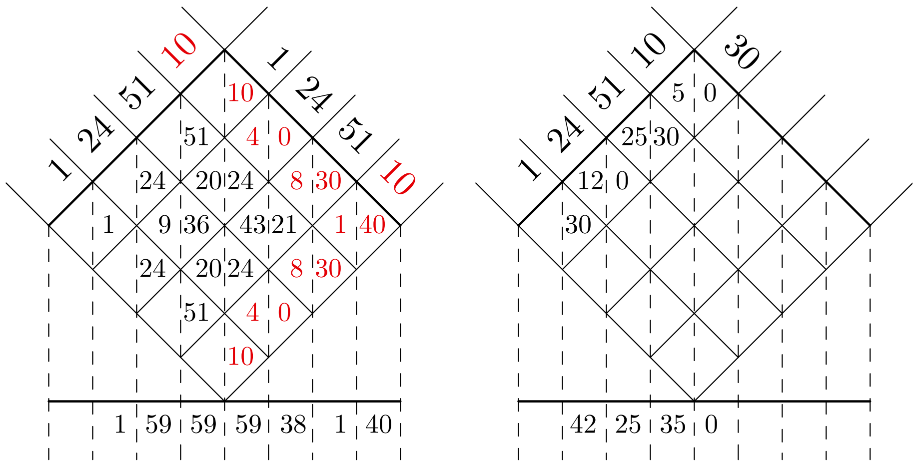


Figure 10: *Approximation of* $\sqrt{2}$ *and of* $\frac{\sqrt{2}}{2}$ *correct to* 6 *decimal digits*

The three-place sexagesimal approximation of the $\sqrt{2}$

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.41421296296\ldots$$

is accurate to six decimal places. It is credible that the Babylonian mathematicians worked out this very accurate approximation with the sole ability to multiply and divide integers, which they did in base 60. This is a hypothesis which is strongly supported by the archaeological artefacts.

The YBC7289 tablet (Figure 2) also gives an example where one side of the square is half the original, i.e., $30^\circ$, and the length of the diagonal is approximately $42''25'35^\circ$; see Figure 10. This gives us the elegant approximation

$$\frac{\sqrt{2}}{2} \approx \frac{42}{60} + \frac{25}{60^2} + \frac{35}{60^3}\,,$$

which may be obtained by halving the approximation for $\sqrt{2}$ above.

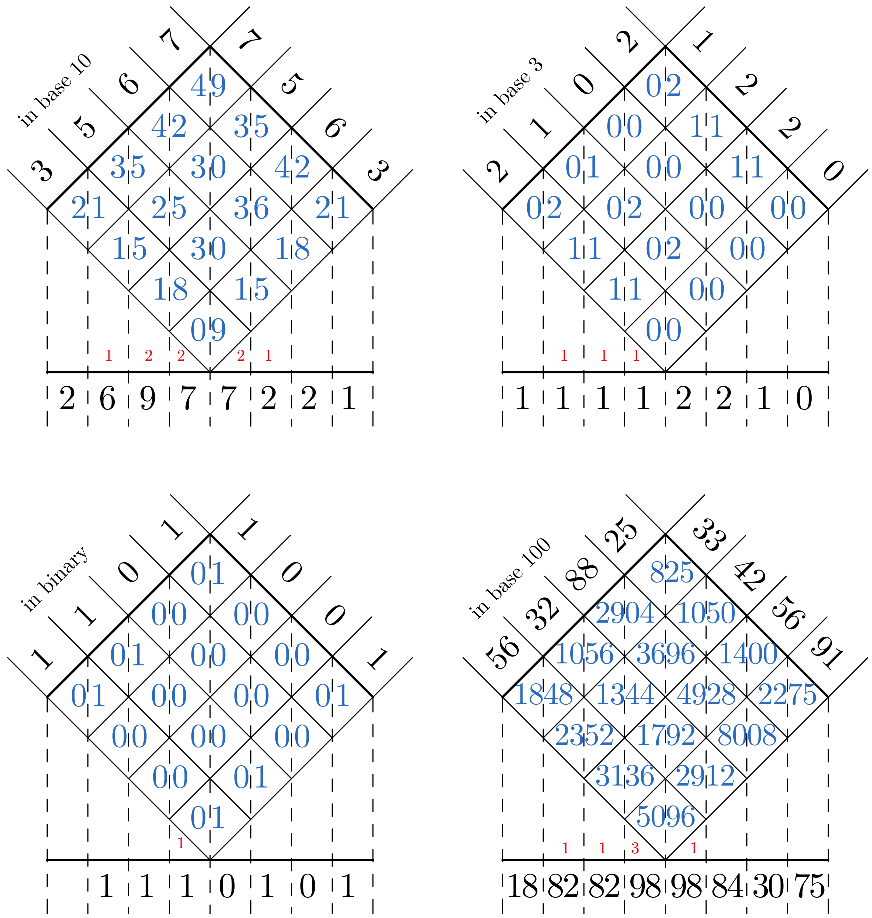


Figure 11: *Examples of the multiplication array in various base systems*

# References


[1] Tablet YBC 7289 (c. 1800–1600 BCE). Credit: Bill Casselman.

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