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Vieta jumping: visualization and intuition

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1 Introduction

In 1988, participants in the International Mathematical Olympiad synchronously opened their test booklets, revealing the problems for the second day of the contest. They were given four and a half hours to solve three problems, the hardest of which was Problem 6. This problem, which has since been discussed in Numberphile's video "The Legend of Question Six", was one of the hardest problems the contest had ever seen. Its average score of 0.634 points out of 7 was the lowest average score on a Problem 6 until 2002 [1].

2 The hardest Olympiad problem (at the time)

In this article, we will look at the concept of Vieta Jumping, which is best encapsulated in the 1988 International Mathematical Olympiad's Problem 6. We will cover the main prerequisite required, walk through the problem, explore a more intuitive explanation of the solution, graph a visual representation of it, and present several related problems.

Vieta's formulas

The only prerequisite knowledge for the proofs in this article are Vieta's formulas for roots of quadratic equations:

Lemma 1 (Vieta's Formulas for Quadratic Equations)**.**
Let r and s be the roots of the real quadratic polynomial $ax^2 + bx + c$, where $a \neq 0$. Then

$$r + s = -\frac{b}{a} \qquad \textit{and} \qquad rs = \frac{c}{a}\,.$$

This lemma may be proved by factoring:

$$ax^2 + bx + c = a(x - r)(x - s) = ax^2 + (-ar - as)x + (ars)\,.$$

Since the first and last polynomials are equal, we may equate their terms, and thereby find that $-ar - as = b$ and $ars = c$. Thus, $r + s = -\frac{b}{a}$ and $rs = \frac{c}{a}$.

Note that if the leading coeficient a is 1, then Vieta's formulas simplify to

$$r + s = -b \qquad \text{and} \qquad rs = c\,.$$

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2.1 The Legend

Example 2 (1988 IMO Problem 6)**.**
Let a and b be positive integers such that $ab+1$ divides a^2+b^2.
Prove that $\frac{a^2+b^2}{ab+1}$ is the square of an integer.

We can rewrite the condition in the problem as

$$(ab+1)k = a^2+b^2 \tag{$*$}$$

for some integer k, and without loss of generality we may assume that $a \geq b$. The problem is then to show that k is a perfect square.

To solve the problem, we first use Vieta's formulas to show the following remarkable fact.

Lemma 3 (Vieta jumping)**.** *If $a \geq b$ and k are positive integers satisfying equation $(*)$, then there is a non-negative integer $a' < a$ such that a', b and k also satisfy $(*)$, with a' playing the role of a. Moreover, a' is positive unless $k = b^2$.*

Proof. The equation $(*)$ can be interpreted as a quadratic equation in a with coefficients depending on b and k, which we may rewrite as

$$a^2 - bka + b^2 - k = 0\,.$$

Let a' be the other solution to this quadratic equation. Then we immediately get that a', b and k also satisfy $(*)$, with a' playing the role of a. It remains to show that a' is a non-negative integer, and positive if $k \neq b^2$. We have from Vieta's formulas that

$$a + a' = bk \qquad \textit{and} \qquad aa' = b^2 - k\,.$$

Therefore, since a, b and k are integers, so is $a' = bk - a$. To show that a' is non-negative, note that

$$(1+a)(1+a') = 1 + a + a' + aa' = 1 + bk + b^2 - k = 1 + (b-1)k + b^2 \geq 1 + b^2$$

is positive, where we have used both Vieta formulas in the second equation, and the fact that $b \geq 1$ in the inequality. Since $1+a$ is also positive, so is $1+a'$. As a' is an integer, this implies that a' is non-negative.

Finally, from the second Vieta formula and the fact that a is positive, we see that $a' = 0$ if and only if $k = b^2$. □

This lemma shows that for every pair of positive integers $a \geq b$ such that $(ab+1)$ divides a^2+b^2 with quotient k where $k \neq b^2$, we can use Vieta's formulas to "jump" to another pair of positive integers a', b with $a' < a$ such that $(a'b+1)$ divides $(a')^2 + b^2$ with the same quotient k.

We can now complete the solution to the problem. For a fixed positive integer k, consider the set of all pairs of positive integers (a, b) with $a \geq b$ which satisfy the equation $(*)$. For each such pair, define the index of the pair to be the sum $a+b$, which is a positive integer. If the set of solutions is nonempty, then, by the well-ordering principle, there must exist a solution (a_0, b_0) with smallest index. Choose such a solution.

Now by the previous lemma, we can jump to another solution (a_0', b_0) of $(*)$ with $a_0' < a_0$. Suppose that $k \neq b_0^2$. Then the new pair (a_0', b_0) also consists of positive integers (here, b_0 may be greater than a_0', but we can simply reverse the order in this case). However, since $a_0' < a_0$, the index of (a_0', b_0) is strictly smaller than that of (a_0, b_0), which contradicts the fact that (a_0, b_0) has smallest index among all solutions. Therefore, it must be that $k = b_0^2$, and in particular k is a perfect square.

This argument shows that equation $(*)$ only has a solution if k is a perfect square.

A visual representation

The above solution may seem unmotivated and difficult to visualize, so we also look at a graphical approach to Vieta jumping. A Vieta jumping "path" for the square $k = 4$, starting from the solution $(a, b) = (112, 30)$, is shown in Figure 1.

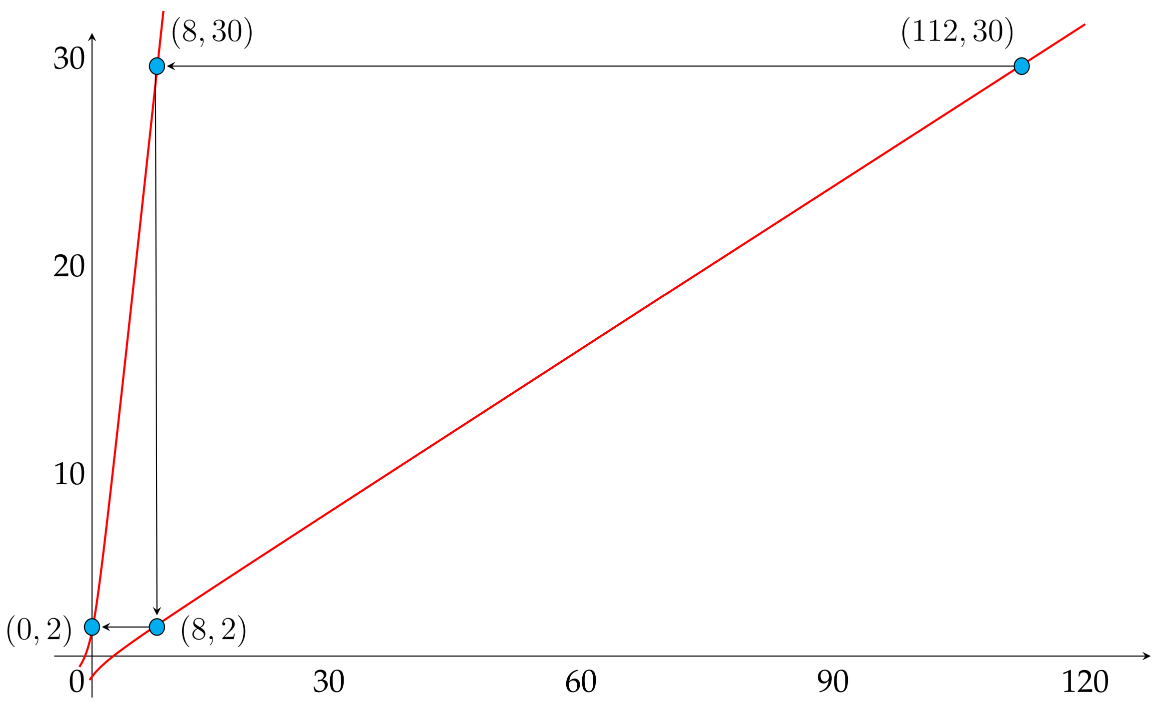

Figure 1: The Vieta jumping path from $(112, 30)$ to $(0, 2)$ on the curve $\frac{x^2+y^2}{xy+1} = 4$.

The image shows the graph of the curve $\frac{x^2+y^2}{xy+1} = 4$, along with some marked points whose coordinates are positive integers. From any such positive integer solution, one can "jump" to another one which shares the smaller coordinate of the first solution - hence the name "Vieta jumping". Along the displayed path, we keep jumping to smaller and smaller solutions until we reach the point $(0, 2)$. At this point, one of the components is 0, so the path is finished.

The reason that k must be a square is that, starting from any positive integer solution for k, we can always "jump" down from that solution until reaching one of the coordinate axes, meaning that we obtain a solution one of whose coordinates is 0. Call the other coordinate of this solution n. Then we would have $k = \frac{n^2+0^2}{0\cdot n+1} = n^2$, so that k is a perfect square. In the above proof, by taking the solution with the smallest index, we consider the solution immediately preceding the jump to the coordinate axis.

The path in Figure 1 does not actually label all positive integer points on the graph, only half of them (since the equation is symmetric in x and y). In Figure 2, the two intertwining paths extending from the origin are shown.

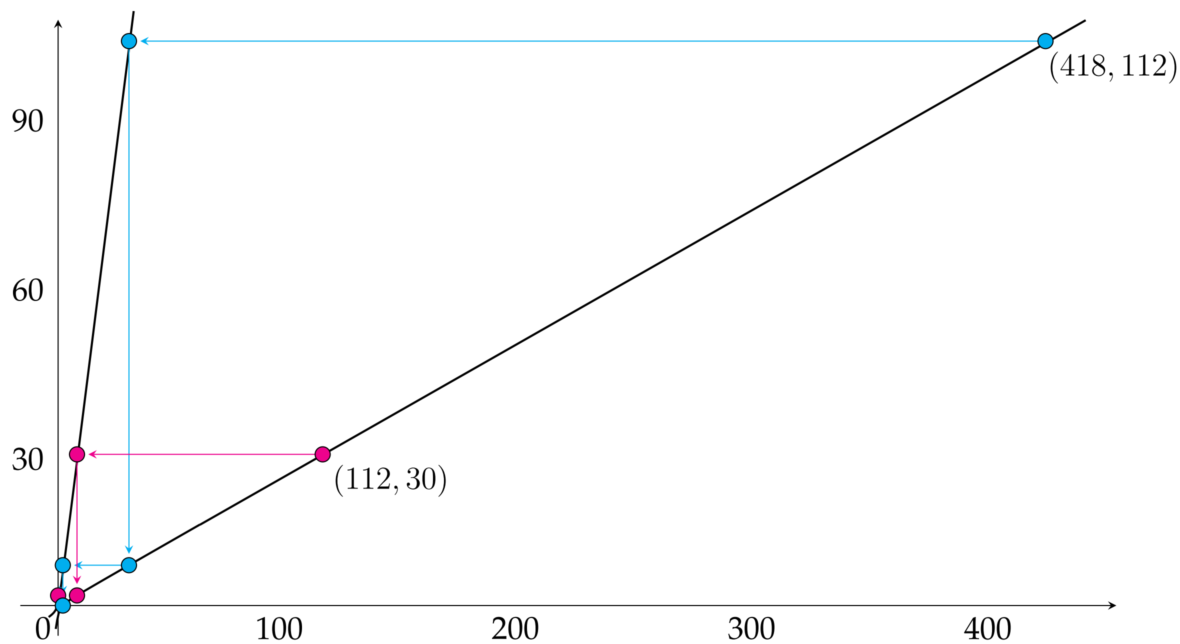

Figure 2: The intertwining Vieta jumping paths from $(418, 112)$ and $(112, 30)$. Each solution to the Diophantine equation lies on the extension of one of these two paths.

3 A similar problem

We now give another example of a problem involving a Diophantine equation which can be solved using the technique of Vieta jumping.

Example 4.
Let a and b be positive integers. Show that if $\frac{a^2+ab+b^2}{2ab+1}$ is an integer, then it is a square.

Again, we rewrite the condition in the problem as

$$(2ab+1)k = a^2 + ab + b^2 \tag{$**$}$$

for some integer k, and we assume that $a \geq b$ are positive integers. We want to show that k is a perfect square.

We have a similar Vieta jumping lemma as before.

Lemma 5. *Suppose that $a \geq b$ and k are positive integers satisfying equation $(**)$. Then there is a non-negative integer $a' < a$ such that a', b and k also satisfy $(**)$, with a' playing the role of a. Moreover, a' is positive unless $k = b^2$.*

Proof. As before, let a' be the other solution to equation $(**)$, interpreted as a quadratic equation in a:

$$a^2 + b(1 - 2k)a + b^2 - k = 0 \,.$$

We have from Vieta's formulas that

$$a + a' = -b(1 - 2k) \qquad \text{and} \qquad aa' = b^2 - k \,.$$

Therefore, since a, b and k are integers, so is $a' = b(1 - 2k)$. To show that a' is non-negative, note that

$$(1 + a)(1 + a') = 1 + a + a' + aa' = 1 - b(1 - 2k) + b^2 - k = (b - 1)^2 + b + k(2b - 1) > 0$$

is positive, where we have used both Vieta formulas in the second equation, and the fact that $b \geq 1$ and $k \geq 1$ in the inequality. Since $1 + a$ is also positive, so is $1 + a'$. As a' is an integer, this implies that a' is non-negative.

Finally, from the second Vieta formula and the fact that a is positive, we see that $a' = 0$ if and only if $k = b^2$. □

Note that this lemma and its proof are identical to the previous example, except for the calculation that a' is non-negative.

The rest of the solution is now also identical to the previous example: for a fixed positive integer k, any solution (a_0, b_0) with minimal index $a_0 + b_0$ must satisfy $k = b_0^2$, or else we could find a solution with smaller index by the Vieta jumping lemma. Therefore, there can only be a solution if k is a square.

4 A less standard example

Not all Vieta Jumping problems follow the same "given that $\frac{x}{y}$ is an integer, prove that it is a square" formula. Some problems, such as the following modified Turkey Team Selection Test problem, stray from the typical Vieta jumping format.

Example 6 (1994 Turkey TST Problem 6, modified)**.**
Determine all positive numbers $\frac{a^2+b^2+3}{ab}$ for which a and b are also positive integers.

This time the condition in the problem is

$$abk = a^2 + b^2 + 3 \tag{$***$}$$

for some integer k, and again we assume without loss of generality that $a \geq b$. Rather than showing k is a perfect square, this time we want to find all possible values of k.

Note that we cannot have $a = b$ unless $a = 1$, because otherwise

$$k = \frac{a^2 + b^2 + 3}{ab} = \frac{2a^2 + 3}{a^2} = 2 + \frac{3}{a^2}$$

would not be an integer. So unless $a = b = 1$, we may assume that $a > b$.

We again start with a Vieta jumping lemma, though this time it takes a slightly different form than in the previous examples.

Lemma 7. *Suppose that $a \geq b$ and k are positive integers satisfying equation $(*{*}*)$ and $a \geq 3$. Then there is a positive integer $a' < a$ such that a', b and k also satisfy $(*{*}*)$, with a' playing the role of a.*

Proof. Again writing $(*{*}*)$ as a quadratic equation in a,

$$a^2 - abk + b^2 + 3 = 0\,,$$

let a' be the other solution. Then Vieta's formulas give

$$a + a' = ak \qquad \text{and} \qquad aa' = b^2 + 3\,.$$

Then from the first Vieta equation we see that $a' = ak - a$ is an integer, and from the second Vieta equation we see that $a' = (b^2 + 3)/a$ is positive. Finally, since $a \geq 3$ (and in particular we have $a > b$), it follows from the second Vieta equation that

$$aa' = b^2 + 3 \leq (a-1)^2 + 3 = a^2 - 2a + 4 < a^2\,,$$

where the first inequality comes from $b < a$ and the second inequality comes from $a \geq 3$. Dividing the above inequality by a, we get $a' < a$. □

Now fix a value of k, and as before consider a positive integer solution (a_0, b_0) for $(*{*}*)$ with minimal index $a_0 + b_0$. It follows from the Vieta jumping lemma that a_0 must be less than 3, because otherwise we could replace a_0 with a_0' and get a solution with smaller index.

Therefore, the only possibilities are $a_0 = 1$, in which case b_0 must also be 1; or $a_0 = 2$, in which case again $b_0 = 1$ (since $b < a$ whenever $a > 1$). Since $b_0 = 1$ either way, these two possibilities give the values

$$k = \frac{a_0^2 + b_0^2 + 3}{a_0 b_0} = \frac{a_0^2 + 4}{a_0} = 5, 4$$

for $a_0 = 1$ and $a_0 = 2$, respectively. Therefore, 4 and 5 are the only possible values for k in a positive integer solution of $(*{*}*)$.

5 Problems

Here are additional problems that can be solved using the method of Vieta jumping.

Problem 1 (1988 IMO Problem 6, Generalization)**.**
Let x, y and z be positive integers such that $\frac{x^2+y^2}{xyz+1}$ is an integer. Prove that it is also a square.

Problem 2. *Suppose that x and y are positive integers such that x^2+xy+y^2 divides $xy+1$. Prove that $\frac{x^2+xy+y^2}{xy+1}$ is a square.*

Problem 3. [2] *Suppose that x and y are positive integers such that $\frac{x^2+y^2}{xy-1}$ is an integer. Prove that $\frac{x^2+y^2}{xy-1}=5$.*

Problem 4. [3] *Let n be a positive integer such that there exist positive integer solutions (x, y, z) to the equation $\frac{x^2+y^2+z^2}{xyz+1}=n$. Prove that n can be written as the sum of two squares.*

Problem 5 (2007 IMO Problem 5)**.**
Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

6 Conclusion

In this article, we explored applications of the Olympiad method of Vieta Jumping, an important tool which can help solve difficult number theoretic problems impervious to other methods. However, its uses extend beyond just competitions. Vieta jumping also has important implications in the analysis of Diophantine equations (equations of integers), with one example being the Markov equation $x^2+y^2+z^2=3xyz$. Here, Vieta jumping can be used to prove that the set of solutions form a structure called Markov's tree [4]. Further research could investigate applications of this method in the field of Diophantine equations.

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