Parabola Volume 60, Issue 1 (2024)

Almost-linear approximation of $\ln(1+x)$ valid for $0 \le x \le 1$

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It is a well-known estimate that, for small values of real numbers $x \ge 0$ much smaller than 1, the linear function x approximates $\ln(1+x)$ as

$$\ln(1+x) \ = \ x \ + \ \varepsilon_0(x)\,,$$

where the error term $\varepsilon_0(x)$ is

$$\varepsilon_0(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}x^n}{n} \le 0$$

since the Maclaurin series [1, 2] of $\ln(1+x)$ is

$$\ln(1+x) = x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}x^n}{n}\,.$$

Explicitly, for $0 \le x \le 1$, the error term $\varepsilon_0(x)$ ranges over the interval

$$[-0.3052\ldots,\ 0\,]\ ,$$

with $\varepsilon_0(0) = 0$ and $\varepsilon_0(1) = \ln(2) - 1 \approx -0.3052$. This error term is negligible for $x \ge 0$ close to zero, but, for x *not* close to zero, any linear approximation of $\ln(1+x)$ fails.

It would be quite useful if this easy approximation for the natural log function held on all of $[0,1]$. Alas, it does not! The *relative error* $\frac{|\varepsilon_0(x)|}{\ln(1+x)}$ is as great as 44.3% as $x \to 1^-$.

One *can* however prove a similar, almost-linear approximation that is valid on the entire interval. Observe for $0 \le x \le 1$ that $1+x$ is approximated roughly by 2^x for x close to 0 or 1 (the reader can check on a calculator), and $1+x = 2^x$ exactly for $x = 0$ and $x = 1$. Taking logarithms of both sides gives an even better approximation: the following formula interpolates between x when $x \ge 0$ is small, and $x \ln 2$ for $x \le 1$ close to 1, with small relative error on the entire interval.

Theorem 1. *For $0 \le x \le 1$, we have the estimate*

$$\ln(1+x) \ = \ x(\ln 2)^x \ + \ \varepsilon(x),$$

with error $\varepsilon(x) \le 0$ such that $\varepsilon(0) = \varepsilon(1) = 0$, and maximum magnitude $|\varepsilon(x)| = 0.0115\ldots$ at $x = 0.6081\ldots$. The relative error $\frac{|\varepsilon(x)|}{\ln(1+x)}$ does not exceed 2.70% on the interval $[0,1]$.

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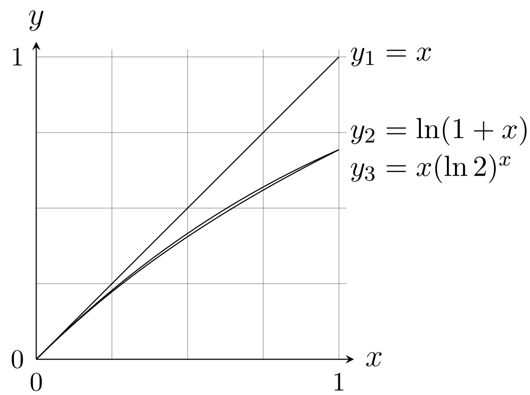
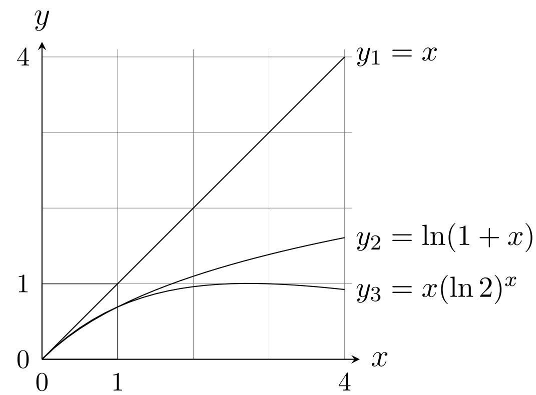
In the first graph below, the reader can see the three curves $y_1 = x$, $y_2 = \ln(1+x)$ and $y_3 = x(\ln 2)^x$ converge as x nears the origin 0; and that y_2 and y_3 essentially coincide on the interval $0 \le x \le 1$, as Theorem 1 states. The second graph illustrates how $y_3 = x(\ln 2)^x$ fails to approximate $y_2 = \ln(1+x)$ when x gets larger than 1, though y_3 still lies far closer to y_2 than does y_1.





To prove Theorem 1, note first that it is clear the error is zero at the endpoints of the interval, since $\ln(1+0) = 0\cdot(\ln 2)^0$ and $\ln(1+1) = 1\cdot(\ln 2)^1$ exactly. The remaining claims in the theorem follow from the first and second derivative tests in calculus, together with the Extreme Value Theorem (see [2]). The author used *Wolfram|Alpha* [3] web-based software to compute the first and second derivatives, and to approximate the critical number and maximum value of $|\varepsilon(x)|$. On the interval $(0,1)$, the relative error $\frac{|\varepsilon(x)|}{1+x}$ achieves its maximum value $0.026954\ldots$, or about 2.70%, at $x = 0.448219\ldots$.

Acknowledgments

Dedicated to my calculus students, past, present and future. Thank you to Maxwell Schneider for advice about numerical approximations.

References

[1] E. Maor, *e: The Story of a Number*, Princeton University Press, 2011.

[2] J. Stewart, D.K. Clegg and S. Watson, *Multivariable Calculus*, Cengage Learning, 2020.

[3] Wolfram Research, Inc., *Wolfram|Alpha Notebook Edition*, Champaign, IL, 2021.