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Solutions 1721–1730

Q1721 Nine people are participating in a "secret Santa" at an office Christmas party. Each brings along a gift which will be passed on to one of the people at the party, so that each person contributes one gift and receives one gift. The name of each person at the party is written on a slip of paper and placed in a bag: each person picks at random a slip from the bag. Then each person gives their gift to the person whose name they have drawn, who then passes it on to the person whose name *they* have drawn: for example, if Andy draws Betty's name and Betty draws Chiara's name, then Andy's gift ends up being given to Chiara. The idea behind passing gifts twice is to ensure that no–one will know who their gift originally came from.

The office newsletter editor, who is not much interested in secrecy, later publishes a report stating that gifts went from Andy to Chiara, from Betty to Harriet, from Chiara to Ivan, from David to Greg, from Elinor to Betty, from Frederica to Andy, from Greg to Elinor, from Harriet to David and from Ivan to Frederica.

Prove that the report is wrong.

SOLUTION We can regard the secret Santa as a "rearrangement" of the gifts, more commonly known in mathematics as a *permutation*. If we write down any person's name, then the person their gift went to, then the person *their* gift went to, and so on, then we must eventually form a *cycle* leading back to the first person; if there remain any people who have not been listed, then we can choose one of them and repeat the procedure. In this way, the distribution of gifts can be written as a "product of cycles", and the published information claims that in the present case, the distribution is

$$\big(\,A\;C\;I\;F\,\big)\big(\,B\;H\;D\;G\;E\,\big)\ .$$

Now consider the permutation which defines one "round" of passing gifts. This also can be written as a product of cycles. Suppose that the product includes a cycle

$$\big(\,x_1\;x_2\;\cdots\;x_{2k}\;x_{2k+1}\,\big)$$

of odd length. Doing this twice, as implemented at the party, gives the cycle

$$\big(\,x_1\;x_3\;x_5\;\cdots\;x_{2k+1}\;x_2\;x_4\;\cdots\;x_{2k}\,\big)\ ,$$

having the same (odd) length, in the "gift distribution" permutation. If there is a cycle

$$\big(\,x_1\;x_2\;\cdots\;x_{2k-1}\;x_{2k}\,\big)$$

of even length, then we get two cycles

$$\big(\,x_1\;x_3\;\cdots\;x_{2k-1}\,\big)\big(\,x_2\;x_4\;\cdots\;x_{2k}\,\big)$$

each of length k. It follows that starting with an odd length cycle cannot produce an even length cycle; starting with an even length cycle can produce an even length cycle (because k might be even), but in that case will always produce two of them. It follows that any result containing an even length cycle once only, such as $\big(A\ C\ I\ F\big)$ in the claimed distribution, is impossible.

Q1722 In Problem 1707, we considered all products of eleven different positive integers having sum 82, and found the greatest common divisor (highest common factor) of all these products. Now change the sum to s, where s is an integer not less than 66. (If $s < 66$, then there is no collection of eleven different positive integers with sum s, and so the problem does not make much sense.) Find the *smallest* value of s for which the greatest common divisor of all the corresponding products of eleven numbers is 1. If $s_{\min}$ is this smallest value and we consider a sum $s > s_{\min}$, then does it necessarily follow that the greatest common divisor of all products of eleven different positive integers with sum s is still 1?

SOLUTION The smallest value of s is $11^2 = 121$. To see this, first note that the smallest possible sum of eleven different positive *odd numbers* is

$$1 + 3 + 5 + \cdots + 21 = 121\ ;$$

so if $s < 121$ and positive integers $x_1, x_2, x_3, \ldots, x_{11}$ add up to s, then at least one of the x_k must be even. Hence, in this case, every product under consideration is even, and the GCD (greatest common divisor) of all the products must be 2, if not more.

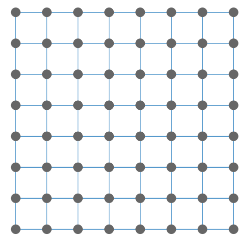
This shows that the required GCD cannot be 1 if the sum s is less than 121; to show that the GCD is equal to 1 when $s = 121$, we follow the ideas of Solution 1707. Consider the following three sums of eleven different positive integers:

$$\begin{aligned}
1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 &= 121\ ;\\
1 + 2 + 4 + 5 + 7 + \ \ 8 + 10 + 11 + 13 + 14 + 46 &= 121\ ;\\
1 + 2 + 3 + 4 + 6 + \ \ 8 + 12 + 16 + 18 + 24 + 27 &= 121\ .
\end{aligned}$$

In the first case, none of the summands has 2 as a factor; so 2 is not a factor of their product, and is not a factor of the GCD of all products. For similar reasons, the second sum shows that 3 is not a factor of the GCD. In the third case, none of the summands has any prime factor greater than 3; so their product has no prime factor greater than 3, and the GCD cannot have such a factor either. We have shown that in the case $s = 121$, the GCD of all products has no prime factor at all, and therefore is equal to 1.

If we consider a sum $s > 121$, then it need not be true that the GCD of all products is still 1. Indeed, let s be an even number greater than 121. Any eleven integers with sum s must include at least one even number (because the sum of eleven odd numbers is odd); therefore, by the same argument as we used in the first paragraph of this solution, the GCD of all products of eleven integers adding up to s must be at least 2.

Q1723 In how many ways can one select 5 points from the 64 shown, such that at least three of the chosen points lie in a straight line? Here, a "straight line" means one of the horizontal or vertical lines shown in the diagram, and the points in a line *do not* need to be adjacent points on the grid.



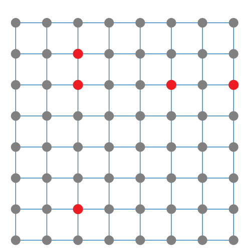
SOLUTION from Hyunbin Yoo, South Korea. Place three points on a line and think about where the remaining two points could be. Depending on whether there are two, one, or zero out of the remaining two points on the same line as the three, the problem can be broken down into three cases.

1. Both points on the same line. If the two remaining points are both placed on the same line as the previously placed three points, then this means that all five points are on a single line. To find the total number of cases, we multiply the number of lines by the number of cases of picking 5 points from a line. This is $16\binom{8}{5} = 896$.

2. One point on the same line. If one of the two remaining points is placed on the same line, there are four points on one line and one not on the line. There are once again 16 lines, and four points to choose out of eight. Finally, choose one point from the 56 points not on the line. We get $16\binom{8}{4}\binom{56}{1} = 62720$.

3. Neither point on the same line. In this case, three points are on a line and two points are outside it: 16 lines, pick three out of eight points, then two out of 56 points. We get $16\binom{8}{3}\binom{56}{2} = 1379840$.

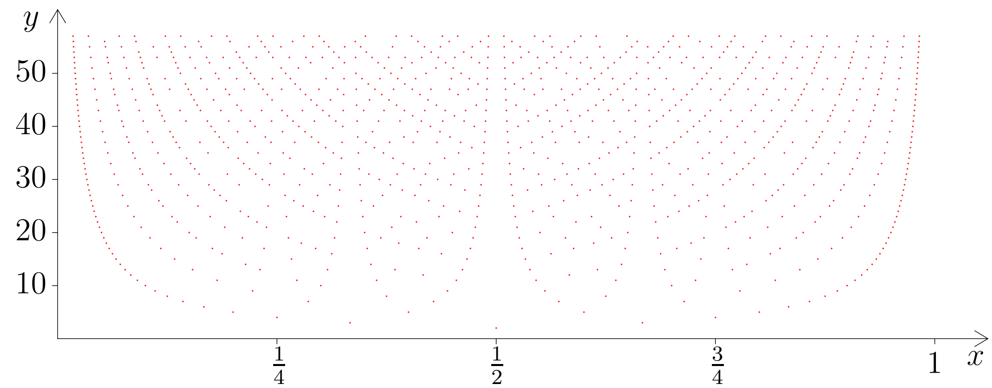
4. Two lines. But wait! It is possible that one set of three points is on a straight line and another on yet another straight line. This is achieved when one of the points is the intersection of one horizontal and one vertical line: the red dots in the diagram give an example.



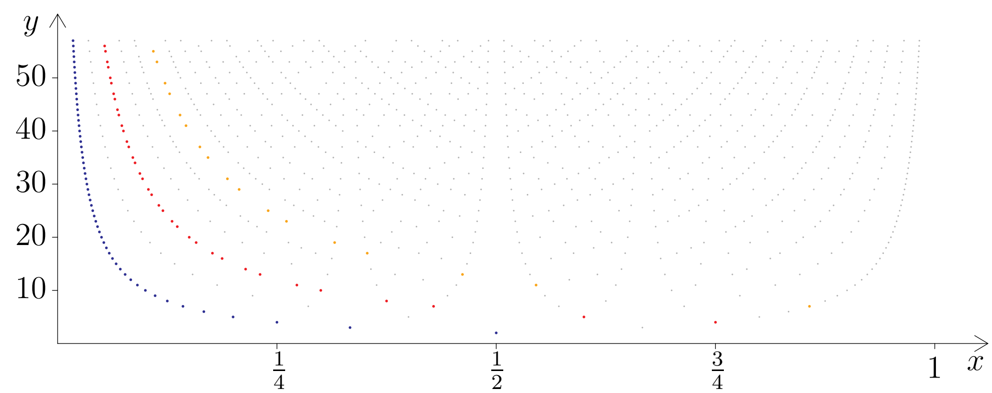
Since these cases are counted twice in Case 3, we need to subtract them once. For each of the 64 points, there is a unique horizontal–vertical line combination. Then we pick two points from each of the two lines; that is, $64\binom{7}{2}\binom{7}{2} = 28224$.

The answer we are looking for is thus $896 + 62720 + 1379840 - 28224 = 1415232$.

Q1724 If x is a rational number, then we define $f(x)$ to be the denominator of x. That is, if $x = p/q$ in lowest terms, then $f(x) = q$. Part of the graph of $y = f(x)$ is shown below. Can you explain the "dotted curves" appearing in the image? Or any other notable features?



SOLUTION First we look at the leftmost "dotted curve", here shown as a sequence of blue dots.



These are the points on the graph corresponding to fractions x with numerator $p = 1$. In this case, we have

$$x = \frac{1}{q} \quad \text{and} \quad y = f(x) = f\left(\frac{1}{q}\right) = q = \frac{1}{x}\,.$$

That is, all of these points lie on the hyperbola $y = 1/x$, which explains why they give the appearance of a smooth curve. The "curves" above this one are formed by fractions x with numerators $2, 3, 4, \ldots$. Likewise, the points on the rightmost "dotted curve" correspond to $p = q - 1$; we have

$$x = \frac{q-1}{q}\,, \quad y = f\left(\frac{q-1}{q}\right) = q = \frac{1}{1-x}\,,$$

and these are therefore points on the curve $y = 1/(1-x)$, a hyperbola with asymptote $x = 1$. The two "dotted curves" forming a "tunnel" above the point $x = \frac{1}{2}$ (and other tunnels elsewhere) can be explained similarly: we invite readers to fill in the details.

One further observation: notice that, for example, the third "curve" on the left (red in the diagram) appears to have gaps in it. This is the curve corresponding to numerator 3; that is, $x = 3/q$. However, if q is a multiple of 3, then we will have to cancel the 3 before computing $f(x)$. For instance, if $q = 12$, then $x = \frac{3}{12}$ does not have denominator 12; rather, we write $x = \frac{1}{4}$ and the denominator is 4. Therefore, this point is "missing" from the $p = 3$ curve and belongs to the $p = 1$ curve instead; the same applies to every third point. For similar reasons, the $p = 6$ curve (orange) includes only the first and fifth out of every six consecutive points, missing the second, third, fourth and sixth; and likewise for other numerators.

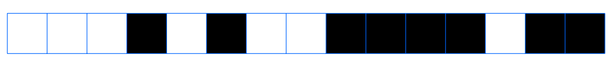
Q1725 Let n be a positive integer. Prove that n^2 has no factor m in the integer interval $n < m \leq n + \sqrt{n}$.

SOLUTION Let m be a factor of n^2 greater than n; then we can write $m = n + a$ where a is a positive integer. Now m is a factor of n^2 by assumption; also, m is a factor of $(n + a)(n - a) = n^2 - a^2$; therefore, m is a factor of the difference $n^2 - (n^2 - a^2) = a^2$. Therefore, we have

$$a^2 \geq m > n ,$$

so $a > \sqrt{n}$ and $m > n + \sqrt{n}$. Therefore, n^2 has no factor m for which $n < m \leq n + \sqrt{n}$.

Q1726 Given a positive integer n, we seek sequences $a_1, a_2, \ldots, a_k$ of one or more positive integers for which it is possible to arrange a row of black and white squares such that the black squares occur in blocks of length $a_1, a_2, \ldots, a_k$, in that order from left to right, and there is at least one white square between adjacent blocks. For example, if $n = 15$, then one of the possible sequences is $1, 1, 4, 2$, as shown by the following row.



For a given positive integer n, in how many ways can this be done?

This problem was inspired by the "nonograms" puzzle which can be found on various websites.

SOLUTION Let a_n be the number of valid sequences for a specified integer n; let $n \geq 3$ and consider a sequence $a_1, a_2, \ldots, a_k$. There are three possible cases.

- Suppose that $a_1 > 1$. Then the sequence is valid for n if and only if the sequence $a_1 - 1, a_2, \ldots, a_k$ is valid for $n - 1$. This is because we can remove a black square from the leftmost block in a row of n squares to create an allowable option for $n - 1$ squares, or, conversely, add an extra black square to the leftmost block in a row of $n - 1$ squares to create an allowable option for n squares. So the number of options in this case is a_{n-1}.
- Suppose that $a_1 = 1$ and the sequence contains further elements (so, $k \geq 2$). Then $a_1, a_2, \ldots, a_k$ is valid for n if and only if $a_2, \ldots, a_k$ is valid for $n - 2$, since we can remove a black followed by a white square from a suitable row of n squares, or, conversely, insert a black and a white into a row of $n - 2$ squares. So the number of options in this case is a_{n-2}.

- The only remaining option is that the sequence consists of a 1 and nothing else; and this is a valid sequence.

Therefore, we have

$$a_n = a_{n-1} + a_{n-2} + 1 \ .$$

Writing this in the form

$$(a_n + 1) = (a_{n-1} + 1) + (a_{n-2} + 1)$$

shows that it is (essentially) the celebrated Fibonacci recurrence. It is easy to show that $a_1 + 1 = 2 = F_3$ and $a_2 + 1 = 3 = F_4$, so for all n we have $a_n + 1 = F_{n+2}$. Thus, for any given n, the number of allowable sequences is

$$F_{n+2} - 1 \ ,$$

where F_k is the kth Fibonacci number.

Q1727 Is it possible to find a square number beginning with any given sequence of digits?

SOLUTION Suppose the given digits form the integer a. Then we want a square whose digits are those of a, followed by (say) k further digits. That is, we want

$$a10^k \le s^2 < (a+1)10^k \ . \qquad (*)$$

Suppose there is no such s, and let t^2 be the largest square less than $a10^k$. Then the next square must be at least $(a+1)10^k$, and we have

$$t^2 \le a10^k - 1 \qquad \text{and} \qquad (t+1)^2 \ge (a+1)10^k \ .$$

Subtracting these inequalities gives $2t \ge 10^k$; together with $t^2 < a10^k$, this yields

$$10^k < 4a \ .$$

So, if we choose k such that $10^k \ge 4a$, then this is impossible; there must be an integer s satisfying $(*)$, and so there must be a square beginning with the digits of a.

Specifically, given a, let k be the number of digits in $4a$, and let the integer s be $\sqrt{a10^k}$, rounded upwards. Then

$$a10^k \le s^2 \ .$$

Moreover,

$$s^2 < \left(\sqrt{a10^k} + 1\right)^2 = a10^k + 2\sqrt{a10^k} + 1$$

and

$$\left(2\sqrt{a10^k}\right)^2 = 4a10^k < 10^{2k} \ ,$$

so

$$s^2 < a10^k + 10^k = (a+1)10^k \ ,$$

and therefore s^2 is a square which begins with the digits of a.

Q1728 Show that if x, y, z are positive real numbers and $xyz = 1$, then

$$x^{1/2} + y^{1/4} + z^{1/6} \geq 2^{2/3}3^{1/2} .$$

SOLUTION We write the left–hand side of the given inequality as a sum of six terms, and then use the Arithmetic–Geometric Mean Inequality:

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} \geq (a_1a_2a_3a_4a_5a_6)^{1/6} .$$

Thus,

$$\begin{aligned} x^{1/2} + y^{1/4} + z^{1/6} &= x^{1/2} + \Big(\frac{y^{1/4}}{2} + \frac{y^{1/4}}{2}\Big) + \Big(\frac{z^{1/6}}{3} + \frac{z^{1/6}}{3} + \frac{z^{1/6}}{3}\Big) \\ &\geq 6\left(x^{1/2}\Big(\frac{y^{1/4}}{2}\Big)^2\Big(\frac{z^{1/6}}{3}\Big)^3\right)^{1/6} \\ &= \frac{6}{2^{2/6}3^{3/6}}\,(xyz)^3 \\ &= 2^{2/3}3^{1/2} . \end{aligned}$$

Solution received from Toyesh Prakash Sharma (who contributed the problem), and from Titu Zvonaru, Comăneşti, Romania.

Q1729 We have n coins, all placed heads up on a table. It is permitted to select any k of the coins and flip them; and to do a similar operation repeatedly. Here, k is a fixed positive integer less than n. The aim is to get all of the coins facing tails up. Prove that this can be done if and only if either n is even or k is odd.

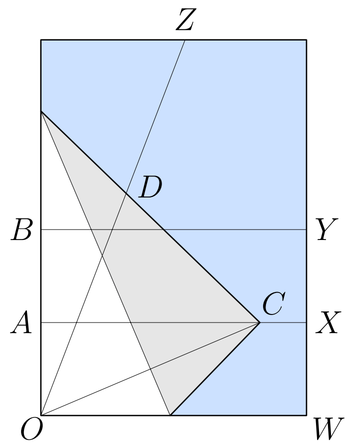
SOLUTION First, suppose that n is even. We can flip any pair of coins, say coins i and j, as follows. Take any set S of $k-1$ coins including neither i nor j; flip S together with i; flip S together with j. Then i and j have been flipped; and any other coin has been flipped either not at all, or twice, which is the same as not being flipped at all. So we now have two coins tails up and the rest heads; we can repeat the procedure to obtain two more coins tails up, and two more, and so on; since n is even, we can make all the coins face tails up.

Secondly, suppose that k is odd. Arrange the coins in a circle. Flip k consecutive coins; then the next k around the circle; and so on; do this n times. This means we have performed nk flips, going all around the circle k times; so every coin has been flipped k times. Since k is odd, every coin is now tails up.

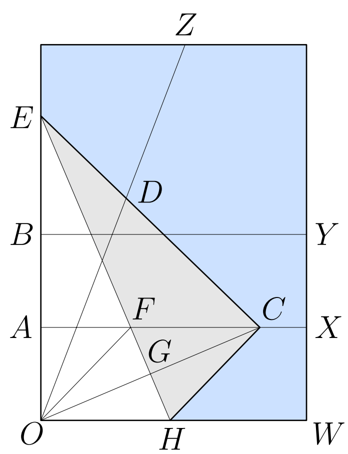
It remains to show that the desired outcome *cannot* be achieved if n is odd and k is even. If it were possible to reach a state in which all coins were tails up, then each would have been flipped an odd number of times (not necessarily the same number of times for each coin). The total number of flips would then be the sum of n odd numbers, that is, an odd number of odd numbers, which is odd; and so it cannot be the result of repeatedly flipping an even number of coins.

Q1730 It is well known that to trisect an arbitrary angle, using ruler and compasses in the classically permissible manner, is impossible. However, the job can be done by origami!

Let OW be one side of a rectangular sheet of paper, and make a fold OZ so that $\angle WOZ$ is the angle we wish to trisect. Make two equally spaced folds AX and BY parallel to OW, as shown in the diagram. Fold the corner O back into the page in such a way that O lies on the line AX, at a point we call C, and B lies on OZ, at a point we call D. Prove that $\angle WOC$ is one third of $\angle WOZ$.



SOLUTION Label points as in the question; also, let E, F, G, H be the points at which the fold meets OA, AX, OC, OW respectively, and draw the line OF. Let $\angle WOC = \alpha$; we shall prove that angles $\angle COF$ and $\angle FOZ$ are also equal to α, so that OC and OF are the trisectors of $\angle WOZ$.



First, we note that $\triangle OHE$ is folded on top of $\triangle CHE$, so

$$|CH| = |OH| ;$$
$$|CG| = |OG| ;$$
$$|CF| = |OF| .$$

Also,

$$|CD| = |OB| = 2|OA| \quad \text{and} \quad \angle HCE = \angle HOE = 90^\circ .$$

Since OC is a transversal of the parallel lines OW and AX, we have

$$\angle FCO = \angle WOC = \alpha\,;$$

and since $|CF| = |OF|$, triangle CFO is isosceles; therefore,

$$\angle FOC = \angle FCO = \alpha\,,$$

and this was the first thing we wanted to prove.

Next, triangle CHO is isosceles, so $\angle OCH$ is also equal to α; hence, $OF \parallel HC$; we already know that $OH \parallel FC$ and $|CH| = |OH|$; so $OHCF$ is a rhombus and its diagonals are perpendicular bisectors of each other. Now consider $\triangle COD$ and $\triangle HCF$. They have equal angles

$$\angle DCO = 90^\circ - \alpha = \angle FHC$$

enclosed by proportional sides,

$$\frac{|CD|}{|CO|} = \frac{2|OA|}{|OC|} = \frac{2|HG|}{|HC|} = \frac{|HF|}{|HC|}\,,$$

the second equality being true because $\triangle OAC$ and $\triangle HGC$ are similar. Thus, triangles COD and HCF are similar; so $\angle COD = \angle HCF = 2\alpha$; hence,

$$\angle FOZ = \angle COZ - \angle COF = \alpha\,,$$

and we are finished.