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Can the chicken cross the road?

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1 Introduction

A little while ago, I was looking at a particular path-finding problem, namely to identify a path through an area while avoiding objects within that area. Working on this problem prompted me to consider whether there was a way to figure out whether a path existed at all, before I bothered to spend time looking for one, let alone trying to optimise it in any way. This led me to rediscover the fascinating world of percolation theory.

This article is a simple introduction to the general ideas with only minimal mathematics. Good further detailed yet accessible discussions appear in [1, 2]. The Wikipedia pages devoted to this topic are also quite informative and a mine of summary data; see for example [3].

The name *percolation theory* arises from the following canonical question. Assume that some liquid is poured on top of some nominally porous material. Will the liquid be able to make its way from hole to hole and reach the bottom? We could also consider such questions as

If a porous rock is submerged under water, will the water reach the centre?

How far from each other should trees be planted in order to minimize the spread of disease or fire?

How infectious does a strain of flu have to be to create a pandemic?

How does the density of certain proteins affect the diffusion of substances through the cell membrane?

How many nodes can be removed before a communications network loses connectivity?

How fractured does a habitat need to be to affect species survival?

At what amount of doping will a non-conducting material conduct?

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The applications go on and on: models of magnetism in physics, colloids, watersheds of landscapes, galaxy formation, market penetration etc. etc.

This is a relatively new field per se, most people citing a series of 1957 papers by Broadbent and Hammersley; see for example [4] as the seminal works, their papers concerning with the design of carbon filters for gas masks. However, many of the theoretical results, as so often happens in applied mathematics, come from previous work in physics with much narrower application.

The range of applications and the mathematical depth of this topic is such that a UNSW Library search still found over 700 academic papers with the word "percolation" in the title in 2022. Many results are quite recent. For example, the proof of percolation for hard disks, which are defined below, apparently first appeared as recently as 2014 [5]. It is still an item of current research and there are still many open problems.

One reason for the interest in this topic is the existence of critical thresholds where the qualitative behaviour of a system changes suddenly with small changes of some underlying system parameter. Such a change in behaviour is often called a "phase transition" in physical systems.

Indeed, in an infinite-sized system, the probability that a connected path exists is either exactly zero or exactly one, and there must be a critical parameter (defined below) threshold below which the probability is always 0 and above which the probability is always 1.

If the size of each step is small compared to the overall object, for example sand grains in a lump of sandstone, then the assumption of infinite size is reasonable. Nevertheless, for finite-sized areas, it has been observed that the probability of an open path can increases sharply from very close to zero to very close to one in a short span of values of the driving parameter. This will be discussed further below.

Percolation theory and applications cover both the discrete and continuous domains. Since it is easier to illustrate the concepts and because there are more analytical results, we will start with the discrete formulation.

2 Discrete percolation

We consider first two dimensions for ease of exposition. Consider some form of grid or tessellation (tiling) of the plane. This may or may not be a regular lattice like a square, triangular, or hexagonal grid; see Figure 1.

In the terminology of the field, the edges are termed "bonds" while the tile areas themselves are called "sites". There are now two types of associated percolation. "Bond" percolation occurs along the links between the nodes (think a pipeline network) while "site" percolation occurs through sites connected by a common edge (think a checkers board). Every bond percolation problem can be realized as a site percolation problem (on a different grid) but the converse is not true.

In bond percolation, there is a probability p that the edge (bond) between two neighbours is "open"; that is, it can be traversed (and, obviously, probability $1 - p$ that it

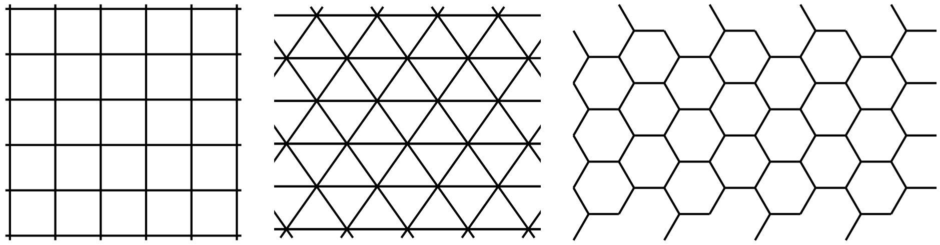

Figure 1: Varieties of grid/lattice: square, triangular and hexagonal.

cannot). Similarly in site percolation, there is a probability p that each site is "occupied" (and $1-p$ that it is "empty"). Each site/edge is assumed to be independent. In applications, p can be a function of some physical quantity, for example temperature or pressure.

The question is, for a given p, what is the probability that an open path[2] exists through the grid? The network is said to percolate if one exists and, thus, a liquid, current or other substance, can be conducted. The answer is trivial for $p = 0$ or $p = 1$. The question is: what happens in between these extremes?

The mathematics becomes very advanced very quickly, but some of the behaviours can be seen in the simplest example, the infinite binary tree. To build this tree, start with a single point V_0. From that point, consider two possible paths to the next level V_1. From these two nodes branch two possible paths to the next level V_2, and so on; see Figure 2.

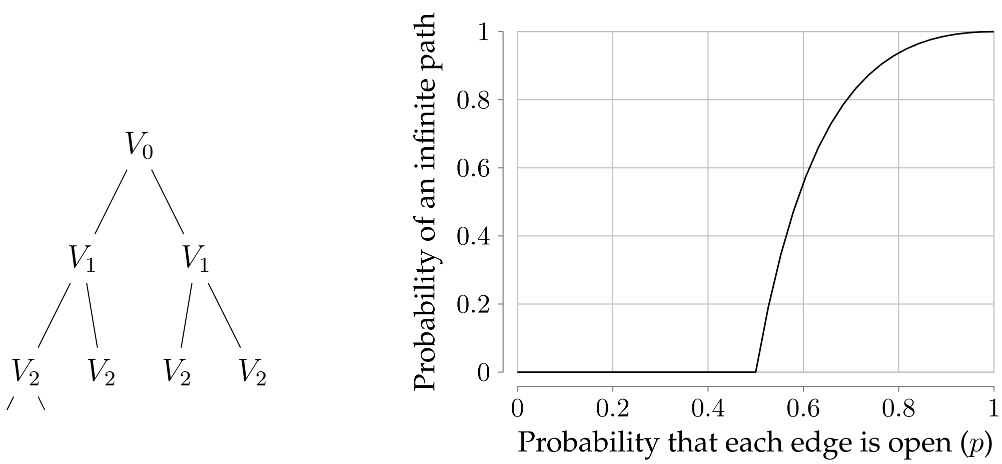

Figure 2: Binary tree. Left: pictorial of tree structure. Right: probability of an infinite path existing plotted against probability that each edge is open.

There is a probability p that each edge is open, and there are $2n$ points at level n that can possibly be reached from V_0. The probability that there is a path to level n, given

[2] Meaning a path, each of whose links is an "open" bond in bond percolation, or via contiguous occupied sites in site percolation.

that there is one from level $n-1$, is

$$P_n = 1 - (1 - pP_{n-1})^2$$

and the probability there is an infinite open path starting from the root of the tree V_0 is

$$P_\infty(p) = \lim_{n\to\infty} P_n \, .$$

It can be shown [1] that

$$P_\infty(p) = \begin{cases} 0 & \text{if} \quad p < \frac{1}{2} \, ; \\ \frac{2}{p^2}\left(p - \frac{1}{2}\right) & \text{if} \quad p \geq \frac{1}{2} \, . \end{cases}$$

Note that there is a *qualitative* as well as quantitative change in behaviour at $p = \frac{1}{2}$, namely from there being no chance of a path through to there being at least some non-zero chance of a path through; see Figure 2. The probability at which this change of behaviour occurs is called the *critical probability*, denoted by p_c, where in this case $p_c = \frac{1}{2}$.

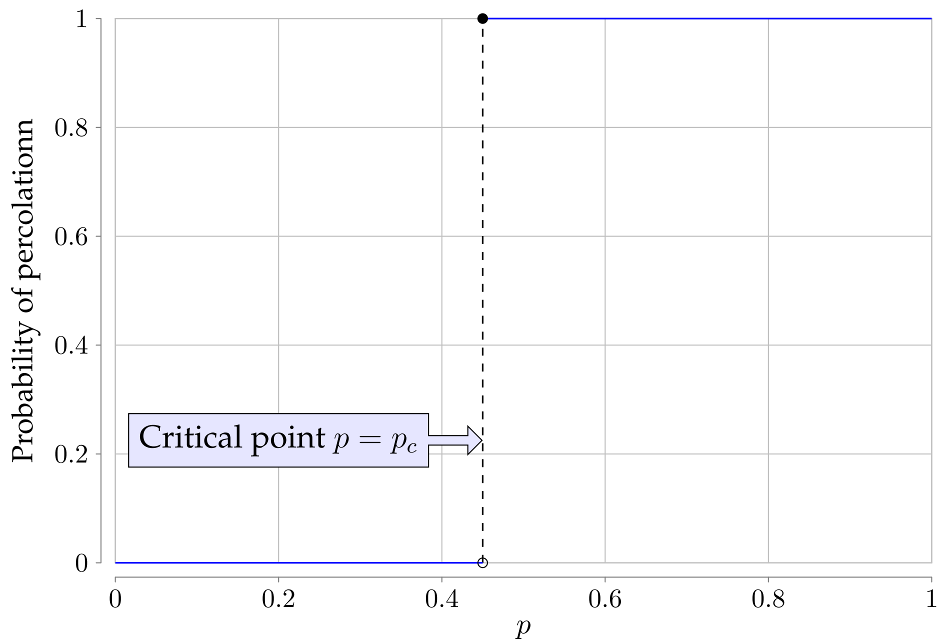

Figure 3: Canonical behaviour of infinite percolation

If V_0 is one of an infinite number of trees[3], then the probability that an infinite path exists in any of the trees is

$$\Pi_\infty(p) = 1 - \lim_{n\to\infty} \left(1 - P_\infty(p)\right)^n \, .$$

[3]Or perhaps one of an infinite number of such trees emanating from a single point.

From the equation for $P_\infty(p)$, this gives:

$$\Pi_\infty(p) = \begin{cases} 0 & \text{if} \quad p < \frac{1}{2}\,; \\ 1 & \text{if} \quad p \geq \frac{1}{2}\,. \end{cases}$$

Thus arises the amazing result that, if each bond is open with a probability less than $p_c = \frac{1}{2}$, then there is 0 probability of a path, technically called a percolating, or a spanning, *cluster*[4] through the trees — and hence, no need to look for one! Another way of expressing this is that if $p < p_c$, then all open clusters are finite with probability 1. If $p \geq p_c$, then there is guaranteed to exist a path through the tree, in the sense that the total probability of that event is exactly 1. Note this result says nothing about finding a path, just that there will be one.

This generic behaviour, illustrated conceptually in Figure 2, applies in generality to all lattices and dimensions[5] The proof of this behaviour comes via something known as the *Kolmogorov Zero–One Law*; see Box 1. An interesting corollary of this law is that one can prove that the probability must be either 0 or 1 without necessarily being able to tell which it is.

Box 1: Kolmogorov Zero-One Law

This deceptively simple mathematical law says that a "tail event" occurs with probability either 1 or 0; i.e., it will either almost surely happen or almost surely not happen. One can prove that this is true for a given event without being able to tell which it is.

The definition of a tail event is quite technical but is defined in terms of infinite sequences of random variables, for example in an infinite sequence of coin-tosses, where heads occurring infinitely many times is a tail event.

Consider an infinite coin tossing with biased coins:

Case 1: prob(head on n'th coin) = $\frac{1}{n}$

Case 2: prob(head on n'th coin) = $\left(\frac{99}{100}\right)^n$

In both cases, the probability tends to 0 as n grows to infinity.

What is probability of tossing an infinite number of heads?

It turns out that the probability is 1 for Case 1 and 0 for Case 2, something I would submit is not obvious at first glance.

This law can be used to prove the well-known thought experiment of whether a monkey typing at random would produce Shakespeare's Hamlet given an infinite amount of time. One can in fact prove that the monkey will not only certainly type Hamlet but will do so infinitely many times. [9]

[4] A "cluster" refers to any chain of connection.

[5] In 3 dimensions, it is easy to envisage some simple lattices such as being cubic; in higher dimensions, not so much, but they exist mathematically.

Critical thresholds have been calculated for a bewildering array of lattices. Some have been determined theoretically; in most cases, only numerical estimates exist. For example, for a square lattice, it can be proven analytically that the bond threshold is exactly $p_c = 0.5$. For site percolation, there is no exact solution, and the threshold has been found numerically to be $p_c = 0.59$ to two significant figures[6]. These two cases are illustrated in Figures 4 and 5.

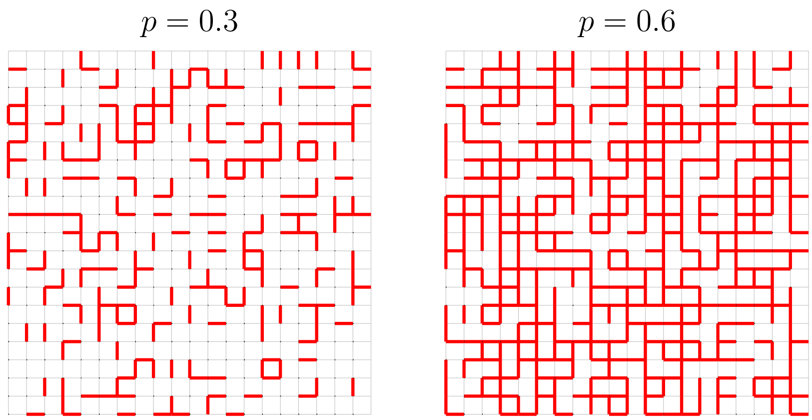

Figure 4: Bond percolation on a square lattice.
Left: $p = 0.3$, no percolation. Right: $p = 0.6$, complete percolation.

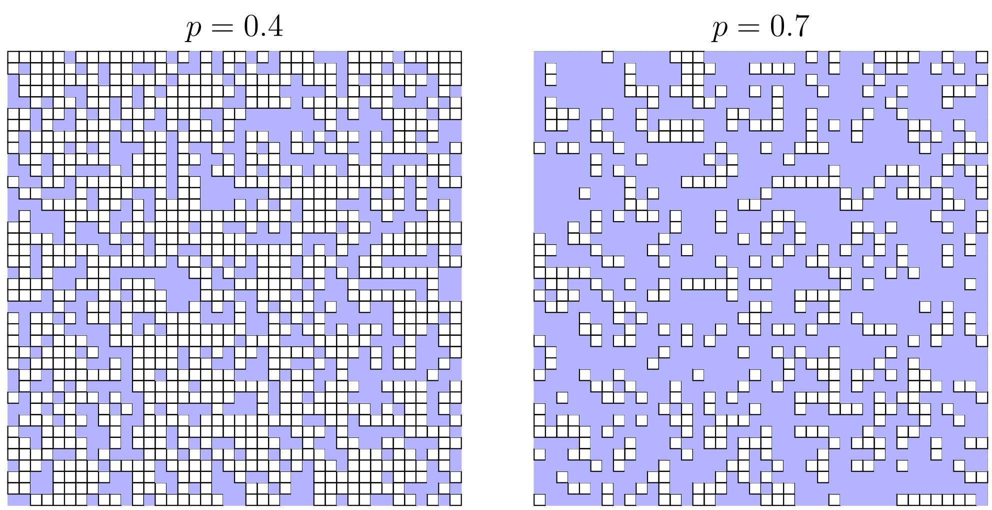

Figure 5: Site percolation on a square lattice. Left: $p = 0.4$, no percolation. Right: $p = 0.7$, complete percolation. If, say, a voltage was placed across the grid and one considers p as the fraction of sites occupied by a conducting substance, then the matrix as a whole becomes conductive for $p \geq p_c$; non-conductive otherwise.

Similarly, for a hexagonal, or honeycomb, lattice, the bond percolation threshold is exactly $1 - 2\sin(\pi/18) \approx 0.653$, and, for site percolation, it is approximately 0.697.

[6]It was long believed that this value should be $\frac{1}{2}$ as well, but it turned out not to be so after careful numerical studies.

There are extra subtleties in considering whether we are discussing percolation specifically in one direction, in either direction, in one and not the other or in both; see for example [6]. However, these change the details of the quantitative results, not the overall behaviour, so will not be discussed here. Importantly, the critical probability/density does not depend on the type of wrapping. Similarly, it does not matter if we are discussing percolating from a source point out to the edges, such as the spread of a fire, for example.

It can also be shown that the infinite open cluster is unique. So, if more than one path is found, then they must be able to be connected.

3 Scaling behaviour and universality

As noted above, percolation in an infinite lattice is equivalent to saying that there exists a cluster of infinite size (the *infinite cluster*). But not all clusters need to belong to the infinite cluster; see for example Figures 4 or 5. Only in the limit as probability approaches 1 will all the bonds/sites be connected and hence all belong to the same cluster. Hence, we can reasonably ask about the behaviour of such statistical quantities as the mean cluster size $S(p)$ and the probability that a bond/site belongs to an infinite cluster $P_\infty(p)$.

These are hard questions that cannot be answered simply. One can infer that, when approaching p_c from below, the average size of the clusters increases as more connections are being formed. On passing p_c, more and more of the links will belong to the infinite cluster; hence, the average size of the clusters, excluding the infinite cluster, will decrease. It turns out that, in the vicinity[7] of p_c, these quantities behave[8] as

$$S(p) \sim |p - p_c|^{-\gamma}$$
$$P_\infty(p) \sim |p - p_C|^{-\beta}$$

where β and γ are the critical exponents for $S(p)$ and $P_\infty(p)$.

Such relationships are called *power laws*. Now, the interesting part is that the value of the critical exponents is *only* a function of the dimensionality of the problem, *not* of the details of the lattice[9]. It also does *not* depend on whether we are considering bond or site percolation.

This power law and universal exponent behaviour is true for several other statistical quantities as well. In two dimensions, the critical exponents are known exactly and have such non-intuitive values as $\frac{43}{18}$ or $\frac{91}{48}$.

It can also be shown that the percolating (infinite) cluster at p_c is a fractal[10] with dimensions related to the critical parameters.

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[7]We will leave this undefined here.

[8]For hopefully obvious reasons, excluding the infinite cluster for $p \geq p_c$

[9]We have already seen that the critical threshold p_c itself *does* depend on the lattice details.

[10]A fractal is a complex geometric shape that is often, but not necessarily, characterised by having a non-integer dimension. They can describe many irregularly shaped objects or spatially non-uniform phenomena in nature, such as coastlines, mountain ranges and snowflakes.

4 Continuous percolation

In this case, consider finite-size objects[11] placed randomly in a space. The randomness can be made more precise in mathematical terms depending on the problem[12], but the general term "randomly" will be sufficient here. The question is whether there exists a continuous path between objects from one side of the space to another.

Numerous objects have been studied in this context: circles, ellipses, squares, rectangles, sticks etc. Objects can be allowed to overlap – such objects are often called "soft" – or not – in which case, they are called "hard". The former usually correspond to regions of interest, such as infection distances or detection ranges. The latter correspond to physical objects such as sand grains or cells. They can also be aligned or have random orientation.

Remarkably, continuous percolation shows the same characteristics as discrete case, namely critical values, phase transitions, scaling universalities and so on. The same scaling behaviour near the critical parameters exist as for discrete percolation – and the critical exponents have the same values!

In the continuous case, the baseline critical parameter taking the role of the bond/site probability is the space filler fraction Φ, given by

$$\Phi = 1 - e^{-\eta}$$

where η is the total relative area occupied by objects.[13] It is easy to see that Φ must be between 0 and 1, as desired.

For discs[14] of radius r, $\eta = \pi r^2 \rho$ where ρ is the object density. For N objects in an area of size A, $\rho = N/A$. Thus, the critical quantity can be expressed in terms of a critical density for given radius of objects or vice versa.

Figure 6 shows an example of two-dimensional percolation using overlapping soft (i.e., overlapping) discs. Discs highlighted in red provide a continuous path of connected discs throughout the space shown.

Arguably, the more common view of percolation is actually the complement of the percolation so far discussed: given objects in the space, what is the probability that there is a path through them? This is certainly true for path-finding applications. In the field, this is usually referred to as *void*[15] percolation. In 2 dimensions, the probability of void percolation is the inverse of the probability of direct percolation[16]; hence,

$$\Phi_c(void) = 1 - \Phi_c(direct) \quad \text{and} \quad \eta_c(void) = \eta_c(dirct)\,.$$

This is *not* true in higher dimensions.

[11]that is, not abstract points of zero dimension.

[12]Two particular physical examples are: that the problem can be initiated by random removal as well as placement, such as trees in a forest, or communication towers; and that the objects can be placed within the constraints of some repulsive potential.

[13]For some objects, such as 1-D sticks, the number n_c of objects of given size per unit is more appro-

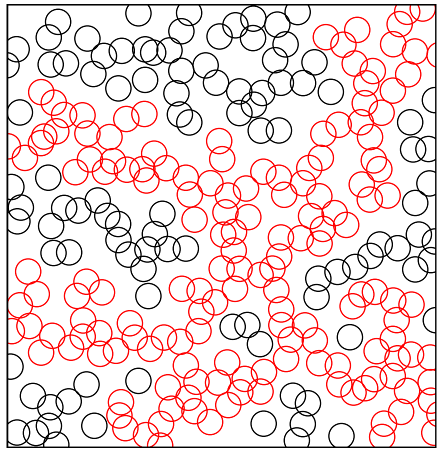

Figure 6: 2D percolation with circular overlapping discs. Discs highlighted in red provide a continuous path of connected discs through the space.

Type of Object	Normal		Void	
	Φ_c	η_c	Φ_c	η_c
Soft Discs	0.68	1.13	$1 - \Phi_c(direct)$ = 0.32	1.13
Hard Discs	0.57	0.85	0.43	0.85
Random ellipses (e = 5)	0.46	0.61	0.54	0.61
Randomly oriented Squares	0.63	0.98	0.32	0.98
Overlapping spheres	0.29	0.34	0.034	3.51
Hard spheres	0.18	0.59		
Randomly aligned cubes	0.21	0.24	0.045	3.1
4 D Overlapping hyperspheres	0.12	0.13	0.0021	6.16

Table 1: Sample critical threshold for various continuous percolation cases

Some critical parameters for continuous percolation are shown in Table 1.[17] Note that the relationship between η and Φ above is only truly valid for overlapping shapes. In particular, for example, the critical value for relative area for overlapping discs is $\eta_c = 1.128$ with a corresponding area fraction $\Phi_c = 1 - e^{-\eta_c} \approx 0.676$. The critical disc radius r_c is then $r_c = \sqrt{\eta_c/(\rho\pi)} \approx 0.6/\sqrt{\rho}$.

It is interesting to compare this result with the simple situation where the discs are placed in a square grid with grid spacing l. In this case, the percolation critical radius r_c is simply $\frac{1}{2}$, the radius at which all the discs are touching by construction. The corresponding number density is $\rho = 1/l^2$, and, hence, $r_c = 0.5/\sqrt{\rho}$.

priate than the relative occupied area. For discs, $n_c = \frac{4}{\pi}\eta_c$.

[14]Which are what circles are normally called in the percolation literature.

[15]Or sometimes "swiss-cheese" (true!)

[16]Consider that if a continuous path of objects exists from, say, top to bottom of a space, then there would be no way to traverse that space between those objects from left to right.

[17]Many, many more are found in [3].

5 Effect of the finite world

Thus far, we have focused on infinite networks looking for paths of infinite length. One could rightly at this point argue that the real world does not have infinite length; nor does it tend to have discontinuities, so it's fair to ask the question: what happens to this behaviour when finiteness is imposed?

It should come as no surprise that the behaviour is smooth for finite systems and approaches the behaviour approximating a discontinuity as the system gets larger. This means that, in principle, we can get connectivity at less than the theoretical percolation threshold or not get it even at a much higher occupancy. To illustrate this behaviour, consider the simple binary tree of Figure 2. It is a simple matter in a spreadsheet to check the probability of percolation for N trees of depth N. The result is shown in Figure 7, together with the result for an infinite system size.

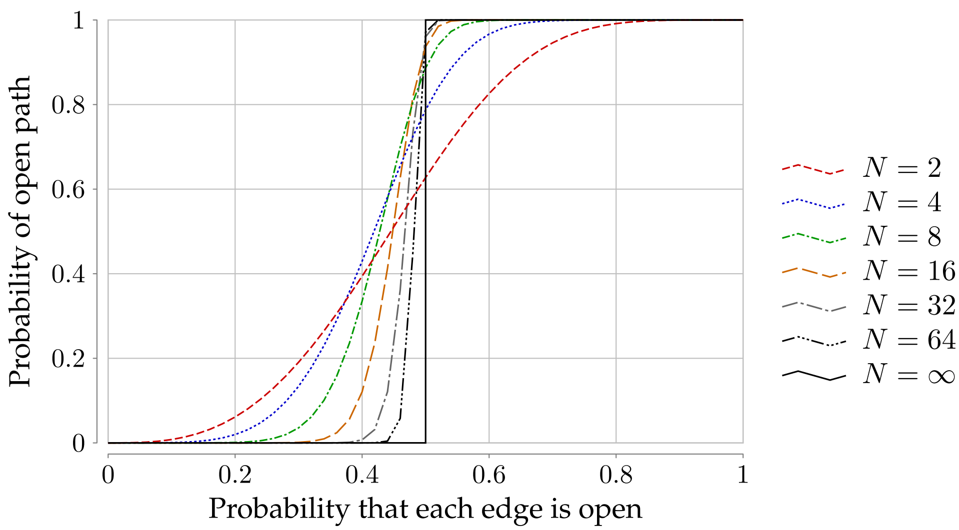

Figure 7: Percolation probability for a binary tree of N levels. As the system size N goes to infinity, the probability tends to a step function around p_c.

The figure illustrates the convergence described above. Importantly, for a "size" of only 64, the jump from 0 to 1 percolation occurs over a span of just 0.05 or so of underlying probability. That is, *a "large" system does not have to be that large to display sudden phase change behaviour*.

The binary tree result in Figure 7 may seem a bit artificial, but graphs similar to Figure 7 have been seen in many applications; see for example [6, 7, 8].

Figure 7 also shows a number of other features of more general behaviour. One is that the curves are not symmetrical around the critical probability. Another is that, except for some special systems, the curves do not all cross each other at a single particular point, although they will tend to do so in the limit of high N.

A final interesting point is that universality of scaling also appears in the finite length case. From reference [7], *"the width of the transition window from ... 0 to ... 1 scales as $L^{-1/\nu}$, where $\nu = 4/3$ is a universal critical exponent for two-dimensional percolation."*.

6 Summary and conclusions

Percolation theory helps us to understand the reasons for which physical processes have sudden changes in behaviour and, in many cases, can predict when those changes will occur. It can be considered in many different mathematical ways, both in discrete and continuous domains. It has interesting mathematical properties such as universality and discontinuities in probability. It also has myriad applications in the real world, from social networks to polymers.

No wonder it is such a hot topic.

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