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Extract from *Vector: A Surprising Story of Space, Time, and Mathematical Transformation*

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This article reproduces pages 11–17 and their endnotes from the recently published book *Vector: A Surprising Story of Space, Time, and Mathematical Transformation*[2] by Robyn Arianrhod (UNSW Press, July 2024). Minor adjustments to the presentation have here been made, most notably the inclusion of the endnotes as footnotes and the use of the *Parabola* citation style.

A cubic conundrum

The Mesopotamians initially had practical problems in mind when they developed the method of solving quadratic equations by geometrically completing the square. (The development of symbolic algebra was still several thousand years in the future, waiting for the work of Thomas Harriot and René Descartes in the early seventeenth century.) Living in a land where water was at a premium, the Mesopotamians' tablets contain many problems relating to canal and reservoir excavations, the capacity of cisterns, the construction and repair of dams and levees, and administrative accounts relating to these tasks – and to solve these problems, these ancient mathematicians had to solve equations relating to areas and volumes. Nearly three thousand years later, al-Khwārizmī, too, focused on similar practical problems, and he used a similar geometrical method of completing the square – and so did other mathematicians right up to the seventeenth century.

The Islamic mathematician Sharaf al-Dīn al-Ṭūsī was one of the earliest to make progress in the search for solutions of cubic equations, in about 1200 CE, but the first to publish correct general cubic algorithms was the Italian mathematician Girolamo Cardano, in his 1545 book *Ars Magna* (*The Great Art*). Like everyone before him, he still wrote his solutions in words (or abbreviations of words) rather than symbols, and he still devised his method geometrically – literally completing a cube in a stunning feat of visualization.

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[2]A review of this book can be found as another article in this issue of *Parabola*.

A mathematical duel, a pesky equation, and an imaginary number

As well as being a talented mathematician, Cardano was a physician, an astrologer, a gambler, something of a philosopher, and a mystic who believed that his best ideas came from a spirit who visited him at night. In the case of cubic equations, however, he received his inspiration from his countryman Niccolò Tartaglia rather than his faithful ethereal advisor. Cardano had heard that Tartaglia had cracked the problem, and he was so intrigued that he badgered him to reveal his method – he even offered to use his connections to put the impecunious Tartaglia in touch with influential people who might pay him for his work on such useful topics as ballistics. Tartaglia finally relented, on condition that Cardano keep the method secret – Tartaglia naturally wanted to publish it himself or, better still, offer it to a future patron.

Some years later, while Tartaglia was still holding onto his secret, Cardano discovered that Scipione del Ferro had also found the solution, before Tartaglia. So, Cardano felt he could break his promise and publish – he always had his eye on a publicity opportunity – but he fully acknowledged both men, and he went beyond them in solving a wider, more general range of equations. Still, Tartaglia was furious – so much so that he challenged Cardano to a public duel, not with swords but with a problem-solving competition. Cardano prudently refused: reputations (and jobs) were easily won and lost in these fiercely competitive Renaissance spectacles. Besides, Tartaglia had already taken on del Ferro's student Antonio Fior, who knew of his teacher's cubic method – and Tartaglia had won that match.

In his book, Cardano explained his general algorithm in a page of ingenious geometrical analogy and then gave specific illustrative examples. This is how he explained his method for solving $x^3 = 6x + 40$ (to use modern notation, which I'll also use to make Cardano's algorithm a little easier to follow; bear with me, even if you just skim through it, because the form of the expression in the last line has surprising relevance to the story of imaginary numbers and, in turn, vectors):

> "*Raise* 2*, one-third the coefficient of* x*, to the cube, which makes* 8*; subtract this from* 400*, the square of* 20*,* [*which is*] *one-half of the constant, making* 392*; the square root of this added to* 20 *makes* $20 + \sqrt{392}$*, and subtracted from* 20 *makes* $20 - \sqrt{392}$*; and the sum of the cube roots of these,* $\sqrt[3]{20 + \sqrt{392}} + \sqrt[3]{20 - \sqrt{392}}$*, is the value of* x*.*"

Phew! You've got to admire his patience in coming up with something so convoluted.[3]

[3] Cardano's underlying algorithm (based on Tartaglia's) for solving an equation of the form $x^3 = cx + d$ is this: choose new variables u, v and set $x = u + v$, $uv = c/3$. Put these into the original equation, and you'll get $u^3 + v^3 = d$; eliminate v and this becomes a quadratic equation in u^3, which can be solved using the quadratic formula. Put this solution for u^3 into $u^3 + v^3 = d$ and solve for v^3. Take the cube roots of u^3 and v^3 to find u, v, and hence $x = u + v$. It's ingenious, and all created without the modern symbolism that makes it easier to keep track of your thought processes. The example I gave, $x^3 = 6x + 40$, and Cardano's algorithm for solving it – together with his geometric completion of the cube – is in Chapter XII of his Ars Magna, reprinted on page 230 in [R. Laubenbacher and D. Pengelley, *Mathematical Expeditions*, Springer, New York, 1999].

The interesting thing, from the point of view of the story of vectors – and of the development of mathematics in general — is what happens when the number under the square root sign in such a solution is negative. That is, when you have an imaginary number such as $\sqrt{-121}$.

The Mesopotamians had ignored negative and imaginary solutions of quadratic equations because they had no relevance to the practical problems they were trying to solve — you can't have negative or imaginary dimensions of fields and canals. Similarly for the Greeks, through to al-Khwārizmī and al-Ṭūsī, and right up until Cardano was forced to wrestle with these "impossible" numbers. He was studying the mathematics of equations simply for its own sake, for the intellectual challenge of it — and he was flummoxed by the fact that if he took the same method he'd used for $x^3 = 6x + 40$ and applied it to $x^3 = 15x + 4$, then the value of x turned out to be

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}\,.$$

Cardano concluded that such a solution was "sophistic," and "as subtle as it is useless" — because aside from the unwelcome $\sqrt{-121}$, he already knew that in fact $x = 4$. He knew this because he would have begun to understand the problem by guessing the solution — something mathematicians have always done. It is especially useful when there isn't a known algorithm for solving a problem, so it is the way ancient algebra began. For Cardano's equation $x^3 = 15x + 4$, you can see the idea by trying a simple possible value such as $x = 3$; comparing each side you see that this is too small, so try $x = 4$. In this case it works straight away, but sometimes you have to try intermediate values. This is still the way mathematicians solve difficult problems "numerically," although they have algorithms (and now computers) to choose their guesses efficiently and exhaustively.

Fifteen years later, around 1560, yet another excellent early modern Italian algebraist, Rafael Bombelli, took another look at Cardano's conundrum. The question was, what did $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$ have to do with it, given that the solution was $x = 4$? After a great deal of thinking, Bombelli suddenly had what he called "a wild thought": what if you could factor $\sqrt{-121}$ like this: $\sqrt{-121} = \sqrt{121} \times \sqrt{-1}$, to get $11\sqrt{-1}$? And could you then find what is nowadays called a "complex" number — a mix of real and imaginary numbers — whose cube is $2 + 11\sqrt{-1}$? Amazingly, with trial, error, and a lot of patience he found that $2 + \sqrt{-1}$ is a solution of $\sqrt[3]{2 + 11\sqrt{-1}}$, as you can see if you multiply out $\left(2 + \sqrt{-1}\right)^3$. Similarly, he found that $2 - \sqrt{-1}$ is a solution of $\left(2 - \sqrt{-1}\right)^3$. Adding these together as in Cardano's solution, you get $x = 2 + \sqrt{-1} + 2 - \sqrt{-1}$, which, seemingly miraculously, gives $x = 4$. Mystery solved!

It was solved only for this special case, though, and when Bombelli knew beforehand that $x = 4$ — he'd had a brilliant insight about manipulating imaginary numbers, but he had no general algorithm. He didn't write his equations in the transparent modern symbolic form I've given here, either — and like Cardano, he disparaged $\sqrt{-1}$ as "sophistic". But he did put this strange number more firmly on the mathematical radar when his book Algebra was published in the 1570s. Little did he or anyone else know back then just how useful it would become.[4]

As for cubic equations, it was Harriot who first found general, *symbolic* algebraic solutions, sometime around 1600 — and with no reference to geometry for his proofs. John Wallis — perhaps the best British mathematician between Harriot's time and Newton's — was one of the few near-contemporaries to recognize Harriot's achievement in liberating algebra from geometry, treating, as Wallis put it[5],

> *"algebra purely by itself, and from its own principles, without dependence on geometry, or any connexion therewith."*

Using algebra to envisage geometry expands not just algebra but geometry, too, and we'll see that these two kinds of maths went hand in hand, each influencing the other, as vectors and tensors emerged. But the first step had been to see, as Harriot and Wallis did, that algebra was a subject in its own right, just as geometry was.

Harriot had taken his lead from the versatile early modern Frenchman François Viète, who had begun to use uppercase letters for unknowns and whose treatise on cubic equations Harriot studied assiduously. Harriot used lowercase letters as we do today, and he used symbols so completely that he became a master of symbolic thinking. One of his insights was to show that polynomial equations can be generated by multiplying their factors — for example, two linear factors generate a quadratic, three give a cubic, four a quartic, and so on. This "factor theorem" may seem obvious now — you may have learned it in a senior high school algebra class — but no one before Harriot had written symbolic equations such as

$$(x-l)(x-m)(x-n)=0\,.$$

Actually, Harriot didn't use separate round brackets for products, but wrote the factors one on top of the other with a square bracket around the group. And he tended to use a rather than x for the unknown, and aa instead of a^2. We owe the notation $x, x^2, x^3, x^4, \ldots$ to Descartes, who published it in 1637, although he still sometimes used xx, and even aa, like Harriot. Either way, what this equation hints at is that a cubic equation has to have three solutions, $x = l$, $x = m$, $x = n$, whether they are positive or negative, real or imaginary. By contrast, Cardano's algorithm had spoken of "the" solution, as if there were only one — which is what you'd expect if you were imagining it in terms of a material cube.[6]

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[4] For instance, Schrödinger's equation describes the dynamics of fundamental particles such as photons, electrons, and other subatomic particles — and it contains i. Electromagnetic waves, too, are easier to handle mathematically using the complex form, so i is behind all sorts of modern technology.

[5] *Wallis on Harriot*: quoted on page 490 of [J.A. Stedall, Rob'd of Glories: The posthumous misfortunes of Thomas Harriot and his algebra, *Archive for History of Exact Sciences* **54** (2000), 455–497].
Harriot first to algebraically (symbolically) solve cubics: The great mathematician Lagrange first made this observation; see page 185 of [M. Seltman, Harriot's algebra: reputation and reality, in *Thomas Harriot: An Elizabethan Man of Science*, ed. R. Fox, pages 153–185, Routledge, London, 2000].

[6] Similarly, a quadratic equation has two solutions, a quartic has four solutions, and so on. [...] An example of Harriot's use of factors and symbols to get complex solutions is found in e.g. British Library Manuscript *6783 folios 157,156.

To see the advantage of Harriot's symbolism, which was not too different from the modern version I'm using here, and just as transparent, consider solving that pesky Cardano equation from Bombelli's starting point of knowing that $x = 4$. Harriot's method suggests that first you write $x^3 = 15x + 4$ as $x^3 - 15x - 4 = 0$. This is just what you would have done in high school, and you'd then divide $x^3 - 15x - 4$ by $x - 4$, to get

$$x^3 - 15x - 4 = (x - 4)(x^2 + 4x + 1)\,.$$

This equals zero when $x = 4$ or when $x^2 + 4x + 1$. You can complete the square to solve the quadratic, to find two additional solutions, $x = -2 + \sqrt{3}$ and $x = -2 - \sqrt{3}$, making a total of three solutions. In this case, all the solutions turn out to be real, and Cardano's complicated expression $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$ doesn't come into it. Or so it seems.... Later mathematicians, however, would connect the historical dots by discovering that in fact complex numbers themselves each have three cube roots. So, the three real solutions of Cardano's pesky equation can be recovered from his algorithm![7]

The power of thinking symbolically

The factor approach is elementary today, but it was a huge breakthrough four hundred years ago. Harriot didn't always use it, and its full, more general implication (the fundamental theorem of algebra) would not be proved rigorously for another two centuries. So, following Viète and Cardano, he also devised a whole list of algorithms for solving various types of quadratic, cubic, and quartic equations. But he was clear about the value of algebraic symbolism[8]:

> "*What need is there for verbose precepts*," he said (for even Viète was wordy), "*when our kind of reduction exhibits all the roots* [*that is, all the solutions*] *directly, not only for this type of equation, but for any other case you like.*"

What he was getting at is that generalization is far easier in symbols than in words. And when you can generalize — when you can see common patterns that apply to an unexpectedly wide range of problems — you can make extraordinary progress in science and technology as well as mathematics. For instance, James Clerk Maxwell was able to show the electromagnetic wave nature of light, and to predict the existence of radio waves, because his mathematical analysis of electromagnetism turned up the very same kind of equation that had been used to describe the wave pattern you see

[7]Following Euler (or figs. 3.4 and 3.6 and related discussion in chap. 3), you can write a complex number $a + ib$ as $r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $r = \sqrt{a^2 + b^2}$ and θ is found from the inverse cosine and sine accordingly. From De Moivre's theorem (or simply from the index laws), the cube root of this number is $\sqrt[3]{re^{i\theta}} = r^{\frac{1}{3}} e^{\frac{i(\theta + 2k\pi)}{3}}$, where $k = 0, 1, 2$ gives the three different roots. Applying this to $\sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} = r^{\frac{1}{3}} e^{\frac{i(\theta + 2k\pi)}{3}} + r^{\frac{1}{3}} e^{\frac{i(-\theta - 2k\pi)}{3}} = 2r^{\frac{1}{3}} \cos\frac{\theta + 2k\pi}{3}$, you get the three solutions of Cardano's equation, $x = 4, -2 + \sqrt{3}, -2 - \sqrt{3}$. It is a bit fiddly, but all the steps use only senior high school or freshman university maths.

[8]*Harriot's quotation*: British Library Additional Manuscript 6783 folio 186.

when you pluck a guitar or violin string. And Emmy Noether brilliantly generalized the relationship between mathematical patterns of symmetry and the conservation of physical quantities such as energy and momentum.

More on these examples later; meantime, Harriot scholar Muriel Seltman sums up neatly both Harriot's importance and the power of algebraic symbolism[9]:

> *"There is a reciprocal relation between symbolism and mathematical thought-processes, and it would be hard to overestimate the effect of Harriot's techniques and clarity of thought expressed in a symbolism that directs what you do visually and therefore makes mathematics accessible in a totally new way. . . The visualizability is obvious but profoundly important. It is now possible to manipulate the symbol as if it were the non-visualizable concept of which the symbol is only the embodiment."*

[9]Seltman's quotation on page 184 of her chapter "Harriot's Algebra" in [R. Fox (ed.), *Thomas Harriot: An Elizabethan Man of Science*, Routledge, London, 2000], my emphasis.