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A non-right analogue of the Kepler triangle

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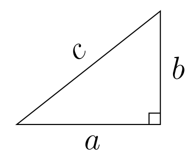
1 Introduction

A *Kepler triangle* is, first of all, a right triangle, and so the side-lengths a, b, c, with $b < a < c$, enjoy the Pythagorean identity:



$$c^2 = a^2 + b^2 \,. \tag{1}$$

In addition, the Kepler triangle has the property that its side-lengths form a geometric progression; for example, with $b < a < c$, this amounts to

$$a^2 = bc \,. \tag{2}$$

Equations (1) and (2) force the ratio of the hypotenuse c to the leg b to be

$$\frac{c}{b} = \varphi \,. \tag{3}$$

Here, φ is the *Golden Ratio*, which is the number

$$\varphi = \frac{1+\sqrt{5}}{2} \,. \tag{4}$$

It is the positive root of $\varphi^2 - \varphi - 1 = 0$, so by re-arranging terms and multiplying by φ^{n-2} for any real number n, we see that φ satisfies the following identities:

$$\varphi^2 = \varphi + 1 \qquad \text{and} \qquad \varphi^n = \varphi^{n-1} + \varphi^{n-2} \,. \tag{5}$$

For more interesting information about the Golden Ratio φ, see [1, 3, 4].

In this article, we provide different descriptions of the Kepler triangle through several equivalent statements. We then show that there is a non-right triangle that exhibits properties similar to those of the Kepler triangle. For the regular Kepler triangle, the side-lengths also satisfy the identity

$$(ac)^2 = (ab)^2 + (bc)^2 \tag{6}$$

which, when re-arranged, becomes $\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{c^2}$. As a result of equations (1) and (6), the Kepler triangle is the only triangle in which both the side-lengths and the reciprocals of

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the side-lengths satisfy Pythagorean identities. Furthermore, if we denote the altitudes from vertices A, B, C by h_a, h_b, h_c respectively, then we always have $\frac{1}{h_c^2} = \frac{1}{h_a^2} + \frac{1}{h_b^2}$ in any right triangle with hypotenuse c, by the Inverse Pythagorean Theorem [**?**]. We will show that the altitudes in the Kepler triangle also satisfy

$$h_b^2 = h_a^2 + h_c^2\,, \tag{7}$$

making the Kepler triangle the only triangle in which the three altitudes and the reciprocals of the three altitudes satisfy Pythagorean identities. At the same time, we have $h_a^2 = h_b h_c$, and so the altitudes in a Kepler triangle form a geometric progression. Not only this, the ratios of the altitudes (or the ratios of the squares of the altitudes) are also golden, in the sense that

$$\frac{h_b}{h_c} = \left(\frac{h_a}{h_c}\right)^2 = \left(\frac{h_b}{h_a}\right)^2 = \varphi\,.$$

Since (2) implies $\left(\frac{1}{a}\right)^2 = \frac{1}{b}\frac{1}{c}$, we can, in conjunction with the re-arranged version of (6), describe the Kepler triangle as a triangle in which the reciprocals of the sides form a geometric progression and the reciprocals of the sides satisfy the Pythagorean identity. Another feature of the Kepler triangle is that if any of the first four equations above hold, then the remaining three become equivalent.

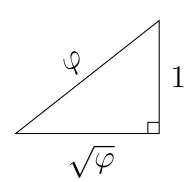
Now let R be the radius of the circumcircle of any triangle ABC. For a right triangle with hypotenuse c, we have $c = 2R$, and so by (1), we can write

$$a^2 + b^2 = 4R^2\,. \tag{8}$$

However, the latter equation is not exclusive to right triangles; in fact, if the side-lengths of a triangle satisfy (8), then either $c^2 = a^2 + b^2$ or $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$. Equation (8) will be used to study a non-right triangle that exhibits properties similar to those of the Kepler triangle.

2 Simple characterizations of the Kepler triangle

This section provides a couple of simple equivalent statements that offer alternative descriptions of the Kepler triangle. Clearly, there is only one Kepler triangle, up to similarity. The diagram to the right shows the side-lengths in a typical Kepler triangle.



Proposition 1. *Let a, b, c be the side-lengths of the Kepler triangle. Then $(ac)^2 = (ab)^2 + (bc)^2$.*

Proof. For the Kepler triangle, we take $c = \varphi$, $b = 1$ and $a = \sqrt{\varphi}$, where φ is the Golden Ratio. Then

$$(ac)^2 = (\sqrt{\varphi} \times \varphi)^2 = \varphi^3$$

$$\text{and} \qquad (ab)^2 + (bc)^2 = (\sqrt{\varphi} \times 1)^2 + (1 \times \varphi)^2 = \varphi + \varphi^2\,.$$

Since $\varphi^3 = \varphi^2 + \varphi$ by (5), it follows that $(ac)^2 = (ab)^2 + (bc)^2$. □

A random right triangle need not satisfy $(ac)^2 = (ab)^2 + (bc)^2$. For example, the very familiar triangle with $c = 5$, $b = 4$ and $a = 3$ doesn't.

Proposition 2. *Let h_a, h_b, h_c be the altitudes from vertices A, B, C in any triangle ABC. Then in a right triangle with side-lengths $b < a < c$, we have that $h_b^2 = h_a^2 + h_c^2$ if and only if $\frac{c}{b} = \varphi$. Thus, the Kepler triangle is the only right triangle in which the three altitudes satisfy a Pythagorean identity.*

Proof. Since $b < a < c$, the altitude h_b is the longest and so the only Pythagorean identity that could hold among the three altitudes h_a, h_b, h_c is the identity $h_b^2 = h_a^2 + h_c^2$.

Also, note that in any right triangle with hypotenuse c and legs a, b, we always have the relationship $\frac{1}{h_a^2} + \frac{1}{h_b^2} = \frac{1}{h_c^2}$. The Kepler triangle is unique among right triangles in that its altitudes are additionally related by the equation $h_b^2 = h_a^2 + h_c^2$.

Suppose that $\frac{c}{b} = \varphi$. Then by Proposition 1, we have $(ac)^2 = (ab)^2 + (bc)^2$. Let R be the radius of the circumcircle of triangle ABC. Then

$$h_a = \frac{bc}{2R}\,, \qquad h_b = \frac{ac}{2R}\,, \qquad h_c = \frac{ab}{2R}$$

and so

$$h_a^2 + h_c^2 = \frac{b^2}{4R^2}\left(a^2 + c^2\right) = \frac{(ac)^2}{4R^2} = h_b^2\,.$$

Conversely, if $h_b^2 = h_a^2 + h_c^2$, then using the expressions for h_a, h_b, h_c above, we obtain

$$(ac)^2 = (ab)^2 + (cb)^2\,.$$

Thus, by Theorem 3 below, $\frac{c}{b}$ is the Golden Ratio φ. □

Theorem 3. *Suppose that the side-lengths a, b, c ($b < a < c$) of a triangle satisfy Equation (1). Then Equations (2), (3), (6) become equivalent. In other words, the following three statements are equivalent in any right triangle (in which $c^2 = a^2 + b^2$):*

1) the side-lengths b, a, c form a geometric progression;

2) the ratio $\dfrac{c}{b}$ is the Golden Ratio φ;

3) $(ac)^2 = (ab)^2 + (bc)^2$.

Proof. As we mentioned previously, condition 3) can be re-written as $\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{c^2}$. Thus, the Kepler triangle is the only right triangle in which the reciprocals of the side-lengths also satisfy the Pythagorean identity.

We establish the above equivalence by showing that 1) $\implies$ 2) $\implies$ 3) $\implies$ 1). So let's suppose that 1) holds; that is, that the sequence b, a, c is geometric. Then $a^2 = bc$. Since $c^2 = a^2 + b^2$, this means that $c^2 = bc + b^2$, as so $\left(\frac{c}{b}\right)^2 - \frac{c}{b} - 1 = 0$. The positive root for the equation $x^2 - x - 1 = 0$ is the Golden Ratio φ, so $\frac{c}{b}$ is this root, and so 2) holds.

Next, we show that 2) $\implies$ 3). Suppose that $\frac{c}{b} = \varphi$. Then $\left(\frac{c}{b}\right)^2 - \frac{c}{b} - 1 = \varphi^2 - \varphi - 1 = 0$, so $c^2 - b^2 = bc$. Since $c^2 = a^2 + b^2$,

$$(ac)^2 = a^2c^2 = a^2(a^2 + b^2) = \left(a^2\right)^2 + (ab)^2 = \left(c^2 - b^2\right)^2 + (ab)^2 = (bc)^2 + (ab)^2\,.$$

Thus, 3) holds. Finally, we show that 3) $\implies$ 1). Suppose then that $(ac)^2 = (ab)^2 + (bc)^2$. Since $c^2 = a^2 + b^2$,

$$a^4 = a^2\left(c^2 - b^2\right) = (ac)^2 - (ab)^2 = (bc)^2\,,$$

so $a^2 = bc$. Therefore, the sequence b, a, c is geometric, and 1) holds. □

Theorem 4. *Suppose that the side-lengths a, b, c ($b < a < c$) of a triangle satisfy Equation (2). Then Equations (1), (3), (6) become equivalent. In other words, the following three statements are equivalent for any triangle in which $a^2 = bc$:*

1) *the triangle is a right triangle with $c^2 = a^2 + b^2$;*
2) *the ratio $\dfrac{c}{b}$ is the Golden Ratio φ;*
3) $(ac)^2 = (ab)^2 + (bc)^2$.

As such, we can describe the Kepler triangle as a triangle in which the side-lengths form a geometric progression and the ratio of the longest side to the shortest side is golden.

Theorem 5. *Suppose that the side-lengths a, b, c ($b < a < c$) of a triangle satisfy Equation (3). Then equations (1), (2), (6) become equivalent. In other words, the following three statements are equivalent for any triangle in which $\frac{c}{b} = \varphi$:*

1) *the triangle is a right triangle with $c^2 = a^2 + b^2$;*
2) *the side-lengths form a geometric progression b, a, c;*
3) $(ac)^2 = (ab)^2 + (bc)^2$.

As such, we can describe the Kepler triangle as a triangle in which the ratio of the longest side to shortest side is golden, and the reciprocals of the sides satisfy the Pythagorean identity (or, the "upside-down Pythagorean identity", in the language of [5]).

Theorem 6. *Suppose that the side-lengths a, b, c ($b < a < c$) of a triangle satisfy Equation (6). Then equations (1), (2), (3) become equivalent. In other words, the following three statements are equivalent for any triangle in which $(ac)^2 = (ab)^2 + (bc)^2$:*

1) *the triangle is a right triangle with $c^2 = a^2 + b^2$;*
2) *the side-lengths form a geometric progression b, a, c;*
3) *the ratio $\dfrac{c}{b}$ is the Golden Ratio φ.*

As such, the Kepler triangle is a triangle in which the reciprocals of the sides satisfy the Pythagorean identity and at the same time form a geometric progression.

3 An obtuse triangle of the Kepler type

In this section, our main objective is to establish analogues of Theorems 3–6) in the setting of non-right triangles satisfying Equation (8). Up to similarity, there is only one such non-right triangle; and the interior angles, determined from Propositions 7 and 8 below, are approximately $26.6°$, $31.7°$, $121.7°$, respectively.

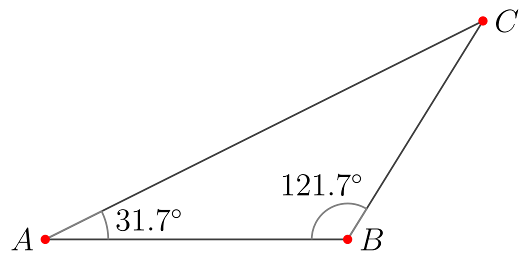


Figure 1: An obtuse, Kepler-type triangle

Proposition 7. *Let a, b, c $(b > a)$ be the side-lengths of a non-right triangle ABC, and let R be the radius of its circumcircle. Then $a^2 + b^2 = 4R^2$ if and only if $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$.*

Proof. Write $R = \frac{c}{2\sin C}$ as per the extended Law of Sines. Suppose that $a^2 + b^2 = 4R^2$. Then by the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C = 4R^2 - 2ab\cos C$, so

$$\cos C\left((a^2 + b^2)\cos C - 2ab\right) = 4R^2\cos^2 C - 2ab\cos C = 4R^2\cos^2 C + c^2 - 4R^2 = 0\,.$$

Therefore, either $\cos C = 0$ or $\cos C = \frac{2ab}{a^2+b^2}$. Since the given triangle is not a right triangle, $\cos C = \frac{2ab}{a^2+b^2}$. However, $\cos C = \frac{a^2+b^2-c^2}{2ab}$, so

$$\frac{2ab}{a^2+b^2} = \frac{a^2+b^2-c^2}{2ab}\,.$$

Therefore,

$$\left(b^2 - a^2\right)^2 = \left(a^2 + b^2\right)^2 - 4a^2b^2 = \left(a^2 + b^2\right)^2 - \left(a^2 + b^2\right)\left(a^2 + b^2 - c^2\right) = (ac)^2 + (bc)^2\,.$$

Conversely, suppose that $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$; then $c^2 = \frac{(b^2-a^2)^2}{a^2+b^2}$, so

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{\left(a^2+b^2\right)^2 - \left(b^2-a^2\right)^2}{2ab\left(a^2+b^2\right)} = \frac{ab}{a^2+b^2}\,.$$

From this, we find that $\sin^2 C = \left(\frac{b^2-a^2}{a^2+b^2}\right)^2$, so

$$R^2 = \frac{c^2}{4\sin^2 C} = \frac{\frac{(b^2-a^2)^2}{a^2+b^2}}{4\left(\frac{b^2-a^2}{a^2+b^2}\right)^2} = \frac{a^2+b^2}{4}\,.$$

Therefore, $4R^2 = a^2 + b^2$ as desired. □

Proposition 8. *Suppose that the side-lengths of a non-right triangle satisfy* $a^2 + b^2 = 4R^2$. *Then* $(ab)^2 = (ac)^2 + (cb)^2$ *if and only if* $\frac{b}{a}$ *is the Golden Ratio* φ.

Proof. Since $a^2+b^2 = 4R^2$, Proposition 7 implies that $\left(b^2-a^2\right)^2 = (ac)^2+(bc)^2$. Suppose that $(ab)^2 = (ac)^2 + (cb)^2$. Then $b^2 - a^2 = ab$, so $\left(\frac{b}{a}\right)^2 - \frac{b}{a} - 1 = 0$, and $\frac{b}{a}$ is therefore the Golden Ratio φ. Conversely, if $\frac{b}{a} = \varphi$, then $\left(\frac{b}{a}\right)^2 - \frac{b}{a} - 1 = \varphi^2 - \varphi - 1 = 0$, so $b^2 - ab - a^2 = 0$. Thus, $(ab)^2 = \left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$. □

The above result, and the next four, are analogues of the ones for the Kepler triangle.

Theorem 9 (Analogue of Theorem 3)**.**
Suppose that the side-lengths of a triangle satisfy $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$.
Then the following statements are equivalent:

1) $\dfrac{b}{a} = \varphi$;
2) $(ab)^2 = (ac)^2 + (cb)^2$;
3) $a, \sqrt[4]{5}c, b$ *is a geometric sequence.*

Proof. The equivalence between 1) and 2) has been shown in Proposition 8 above. We now show that 1) $\iff$ 3). To this end, suppose that $\frac{b}{a} = \varphi$. Let's compute $\sqrt{5}c^2$:

$$\sqrt{5}c^2 = \sqrt{5}\,\frac{(b^2-a^2)^2}{a^2+b^2} = \sqrt{5}\,\frac{a^2b^2}{a^2\left(1+\varphi^2\right)} = \sqrt{5}\,\frac{b^2}{\left(\frac{5+\sqrt{5}}{2}\right)} = \frac{b^2}{\varphi} = \frac{a^2\varphi^2}{\varphi} = a^2\varphi = ab\,.$$

Since $\sqrt{5}c^2 = ab$, it follows that the sequence $a, \sqrt[4]{5}c, b$ is geometric, and so 1) $\implies$ 3). Now suppose that $a, \sqrt[4]{5}c, b$ is geometric; then $\sqrt{5}c^2 = ab$. Since $\left(b^2 - a^2\right)^2 = \left(a^2+b^2\right)c^2$,

$$\sqrt{5}\frac{(b^2-a^2)^2}{a^2+b^2} = ab\,.$$

It follows that $\sqrt{5}\left(b^4 - 2b^2a^2 + a^4\right) = ab\left(a^2+b^2\right)$, and so

$$\sqrt{5}\left(\frac{b}{a}\right)^4 - \left(\frac{b}{a}\right)^3 - 2\sqrt{5}\left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right) + \sqrt{5} = 0\,,$$

or, expressed differently,

$$\left(\frac{b}{a} - \left(\frac{1+\sqrt{5}}{2}\right)\right)\left(\frac{b}{a} - \left(\frac{-1+\sqrt{5}}{2}\right)\right)\left(\sqrt{5}\left(\frac{b}{a}\right)^2 + 4\frac{b}{a} + \sqrt{5}\right) = 0\,.$$

The quadratic factor $\sqrt{5}\left(\frac{b}{a}\right)^2 + 4\left(\frac{b}{a}\right) + \sqrt{5}$ is irreducible and so the only solutions are $\frac{b}{a} = \frac{1+\sqrt{5}}{2}$ or $\frac{b}{a} = \frac{-1+\sqrt{5}}{2}$. Since $b > a$ by assumption, $\frac{b}{a} = \frac{1+\sqrt{5}}{2} = \varphi$. □

Theorem 10 (Analogue of Theorem 4)**.**
Suppose that the side-lengths a, b, c of a triangle form a geometric progression $a, \sqrt[4]{5}c, b$. Then the following statements are equivalent:

1) $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$;
2) $\dfrac{b}{a} = \varphi$;
3) $(ab)^2 = (ac)^2 + (cb)^2$.

Proof. Since the sequence $a, \sqrt[4]{5}c, b$ is geometric, we have $\sqrt{5}c^2 = ab$ and so $5c^4 = (ab)^2$. We show that $1) \implies 2) \implies 3) \implies 1)$. Suppose that $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$. Then

$$\left(b^2 - a^2\right)^2 = c^2(a^2 + b^2) = \frac{ab}{\sqrt{5}}(a^2 + b^2)\,.$$

We encountered this equation before and showed that its solution is when $\frac{b}{a} = \varphi$, proving 2). Next, suppose that 2) is true; that is, $\frac{b}{a} = \varphi$. Then $b = \varphi a = \frac{1+\sqrt{5}}{2}\,a$, so

$$(ab)^2 = 5c^4 = c^2\left(\sqrt{5}\,ab\right) = c^2\left(\sqrt{5}\varphi\right)a^2 = c^2\left(1 + \varphi^2\right)a^2 = c^2\left(a^2 + b^2\right) = (ac)^2 + (cb)^2\,.$$

Finally, we show that $3) \implies 1)$. Suppose that $(ab)^2 = (ac)^2 + (cb)^2$. Then

$$5(ab)^4 = 5c^4\left(a^2 + b^2\right)^2 = (ab)^2\left(a^2 + b^2\right)^2\,,$$

so $\left(a^2 + b^2\right)^2 = 5(ab)^2$. Therefore,

$$\left(b^2 - a^2\right)^2 = \left(a^2 + b^2\right)^2 - 4(ab)^2 = 5(ab)^2 - 4(ab)^2 = (ab)^2 = (ac)^2 + (cb)^2\,. \qquad \square$$

Theorem 11 (Analogue of Theorem 5)**.**
Suppose that $\frac{b}{a}$ is the Golden Ratio φ. Then the following statements are equivalent:

1) $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$;
2) $(ab)^2 = (ac)^2 + (cb)^2$;
3) *the sequence $a, \sqrt[4]{5}c, b$ is geometric.*

Proof. Since $\frac{b}{a} = \varphi$, we have $b^2 - a^2 = a^2(\varphi^2 - 1) = a^2\varphi = ab$. To show that $1) \implies 2)$, suppose that $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$. Therefore, $(ab)^2 = \left(b^2 - a^2\right)^2 = a^2b^2 = (ac)^2 + (bc)^2$. Next up is to show that $2) \implies 3)$. Suppose that $(ab)^2 = (ac)^2 + (bc)^2$; then

$$\sqrt{5}c^2 = \sqrt{5}\,\frac{(ab)^2}{a^2 + b^2} = \sqrt{5}\,\frac{a^2(\varphi^2 a^2)}{(1 + \varphi^2)a^2} = \varphi a^2 = ab\,.$$

This shows that the sequence $a, \sqrt[4]{5}c, b$ is geometric. For $3) \implies 1)$, note that $\frac{b}{a} = \varphi$ and the sequence $a, \sqrt[4]{5}c, b$ being geometric together yield $\left(b^2 - a^2\right)^2 = (ab)^2 = 5c^4$. But then $c^2(a^2 + b^2)$ also equals $5c^4$, following similar calculations as in the proof of the previous theorem. $\square$

Theorem 12 (Analogue of Theorem 6)**.**
Suppose that the side-lengths a, b, c are related via $(ab)^2 = (ac)^2 + (cb)^2$.
Then the following statements are equivalent:

1) $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$;
2) $\dfrac{b}{a} = \varphi$;
3) *the sequence* $a, \sqrt[4]{5}c, b$ *is geometric.*

Summary

The obtuse triangle described above (see Figure 1) satisfies $\left(b^2 - a^2\right)^2 = (ac)^2 + (bc)^2$ and $\frac{b}{a} = \varphi$. This triangle also has other nice properties. Some of these, which we invite the reader to verify, are:

$\sin A \sin B = \sin C$;

$\dfrac{1}{\sin^2 A} + \dfrac{1}{\sin^2 B} = \dfrac{1}{\sin^2 C}$;

the three altitudes satisfy $h_c^2 = h_a^2 + h_b^2$;

the three medians also satisfy $m_c^2 = m_a^2 + m_b^2$;

the nine-point center N lies on AB externally, and $|AN| : |NB| = \varphi^6 : 1$.

Acknowledgements

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