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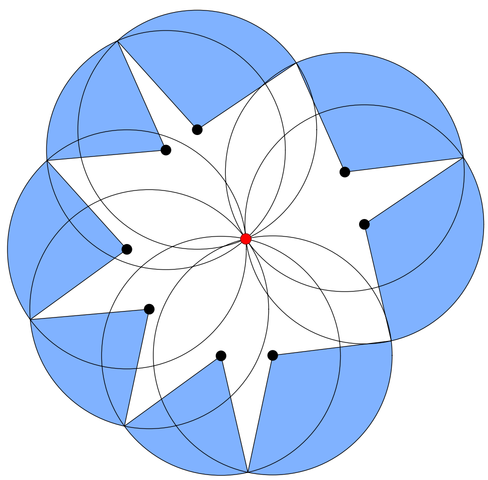
Problems 1741–1750

Parabola would like to thank Les Gordon for contributing Problem 1743.

Q1741 There are n circles of radius 1, all intersecting at a point P: the diagram gives an example with $n = 8$, where P is the red dot in the middle. Lines are drawn from the centre of each circle to its points of intersection (other than P) with the two neighbouring circles. The coloured region lies between these lines and the "external" arcs of the circles. Supposing that none of these "external" arcs contains P, what is the area of the coloured region?



Q1742 Show that if the product of two, three or four consecutive positive integers is increased by 1, then the result is (respectively) never, sometimes or always a square.

Q1743 A pyramid has a rectangular base $ABCD$ and a vertex P (which does not have to be directly above the centre of the base). If AP, BP, CP have lengths $63, 60, 16$, then find the length of DP.

Q1744 Prove that any multiple of 36 can be written as the sum of four different perfect cubes. Note that perfect cubes don't have to be positive: they can be negative, or even zero. For example, $-8 = (-2)^3$ and $0 = 0^3$ are allowable summands.

Q1745 We are given the set of all fractions with numerator 1 and prime denominators:

$$S = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \ldots \right\}.$$

You are allowed to take the difference of any two numbers in S and include it in S; and then to repeat the process, using your newly produced fraction if you wish. For example, you could obtain

$$\frac{1}{2} - \frac{1}{7} = \frac{5}{14} \quad \text{and then} \quad \frac{5}{14} - \frac{1}{7} = \frac{3}{14} \quad \text{and also} \quad \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

and so on. Is it possible to obtain the fraction $\dfrac{1}{2024}$?

Q1746 There are four lines and 2024 red points in a plane. None of the red points lies on any of the four lines; but the lines divide the plane into various regions, each containing an equal number of the red points. Any point where two or more of the lines intersect is coloured green. How many green points are there?

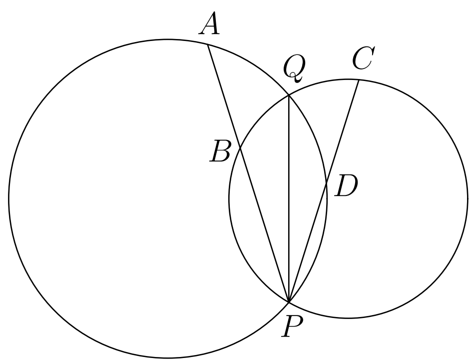
Q1747 Write a_k for the last digit of the sum of the digits of k. This gives a sequence

$$a_1, a_2, \ldots = 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 2, 3, 4, \ldots .$$

Show that this sequence never contains the same digit more than twice consecutively.

Q1748 Three cyclists Andy, Bobby and Cassie ride around a circular track, all in the same direction, at respective speeds of $24, 40, 50$ kilometres per hour. At certain times, all three of them are together. In between two successive triple meetings, how many times are there when two of the cyclists meet?

Q1749 In the diagram, two circles intersect at P and Q; lines through P intersect one of the circles at A and D, and the other at B and C, as shown. If $\angle APQ = \angle CPQ$, then prove that the line segments AB and CD are of equal length.



Q1750 For each letter of the alphabet $A, B, C, \ldots, Z$, we refer to the previous and subsequent letters in alphabetical order as its "neighbours". For example, the neighbours of D are C and E; Z has only one neighbour, Y. We wish to order all 26 letters in such a way that each letter in our list, except the first, is preceded at some stage by at least one of its neighbours. For instance,

$$DCBEAFG \cdots XYZ$$

is an allowable list, but

$$BADECFG \cdots XYZ$$

is not, because D is preceded neither by C nor by E. In how many ways can the alphabet be ordered in this fashion?